1 Foundations of Newtonian gravity

The central theme of this book is gravitation in its weak-field aspects, as described within the framework of Einstein's general theory of relativity. Because Newtonian gravity is recovered in the limit of very weak fields, it is an appropriate entry point into our discussion of weak-field gravitation. Newtonian gravity, therefore, will occupy us within this chapter, as well as the following two chapters.

There are, of course, many compelling reasons to begin a study of gravitation with a thorough review of the Newtonian theory; some of these are reviewed below in Sec. 1.1. The reason that compels us most of all is that although there is a vast literature on Newtonian gravity – a literature that has accumulated over more than 300 years – much of it is framed in old mathematical language that renders it virtually impenetrable to present-day students. This is quite unlike the situation encountered in current presentations of Maxwell's electrodynamics, which, thanks to books such as Jackson's influential text, are thoroughly modern. One of our main goals, therefore, is to submit the classical literature on Newtonian gravity to a Jacksonian treatment, to modernize it so as to make it accessible to present-day students. And what a payoff is awaiting these students! As we shall see in Chapters 2 and 3, Newtonian gravity is most generous in its consequences, delivering a whole variety of fascinating phenomena.

Another reason that compels us to review the Newtonian formulation of the laws of gravitation is that much of this material will be recycled and put to good use in later chapters of this book, in which we examine relativistic aspects of gravitation. Newtonian gravity, in this context, is a necessary warm-up exercise on the path to general relativity.

In this chapter we describe the foundations of the Newtonian theory, and leave the exploration of consequences to Chapters 2 and 3. We begin in Sec. 1.1 with a discussion of the domain of validity of the Newtonian theory. The main equations are displayed in Sec. 1.2 and derived systematically in Secs. 1.3 and 1.4. The gravitational fields of spherical and nearly-spherical bodies are described in Sec. 1.5, and in Sec. 1.6 we derive the equations that govern the center-of-mass motion of extended fluid bodies.

Gravitation rules the world, and before Einstein ruled gravitation, Newton was its king. In this chapter and the following two we pay tribute to the king.

1.1 Newtonian gravity

The gravitational theory of Newton is an extremely good representation of gravity for a host of situations of practical and astronomical interest. It accurately describes the structure

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Table 1.1 Values of ε for representative gravitating systems.

Earth's orbit around the Sun	10^{-8}
Solar system's orbit around the galaxy	10^{-6}
Surface of the Sun	10^{-5}
Surface of a white dwarf	10^{-4}
Surface of a neutron star	0.1
Event horizon of a black hole	~ 1

of the Earth and the tides raised on it by the Moon and Sun. It gives a detailed account of the orbital motion of the Moon around the Earth, and of the planets around the Sun. To be sure, it is now well established that the Newtonian theory is not an exact description of the laws of gravitation. As early as the middle of the 19th century, observations of the orbit of Mercury revealed a discrepancy with the prediction of Newtonian gravity. This famous discrepancy in the rate of advance of Mercury's perihelion was resolved by taking into account the relativistic corrections of Einstein's theory of gravity. The high precision of modern measuring devices has made it possible to detect relativistic effects in the lunar orbit, and has made it necessary to take relativity into account in precise tracking of planets and spacecraft, as well as in accurate measurements of the positions of stars using techniques such as Very Long Baseline Radio Interferometry (VLBI). Even such mundane daily activities as using the Global Positioning System (GPS) to navigate your car in a strange city require incorporation of special and general relativistic effects on the observed rates of the orbiting atomic clocks that regulate the GPS network. But apart from these specialized situations requiring very high precision, Newtonian gravity rules the solar system.

Newtonian gravity also rules for the overwhelming majority of stars in the universe. The structure and evolution of the Sun and other main-sequence stars can be completely and accurately treated using Newtonian gravity. Only for extremely compact stellar objects, such as neutron stars and, of course, black holes, is general relativity important. Newtonian gravity is also perfectly capable of handling the structure and evolution of galaxies and clusters of galaxies. Even the evolution of the largest structures in the universe, the great galactic clusters, sheets and voids, whose formation is dominated by the gravitational influence of dark matter, are frequently modelled using numerical simulations based on Newton's theory, albeit with the overall expansion of the universe playing a significant role.

Generally speaking, the criterion that we use to decide whether to employ Newtonian gravity or general relativity is the magnitude of a quantity called the "relativistic correction factor" ε:

$$\varepsilon \sim \frac{GM}{c^2 r} \sim \frac{v^2}{c^2}, \tag{1.1}$$

where G is the Newtonian gravitational constant, c is the speed of light, and where M, r, and v represent the characteristic mass, separation or size, and velocity of the system under consideration. The smaller this factor, the better is Newtonian gravity as an approximation. Table 1.1 shows representative values of ε for various systems.

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Context is everything, of course. It is now accepted that general relativity, not Newtonian theory, is the "correct" classical theory of gravitation. But in the appropriate context, Newton's theory may be completely adequate to do the job at hand to the precision required. For example, Table 1.1 implies that a description of planetary motion around the Sun, at a level of accuracy limited to (say) one part in a million, can safely be based on the Newtonian laws. The Newtonian theory can also be exploited to calculate the internal structure of white dwarfs, provided that one is content with a level of accuracy limited to one part in one thousand. For more compact objects, such as neutron stars and black holes, Newtonian theory is wholly inadequate.

1.2 Equations of Newtonian gravity

Most undergraduate textbooks begin their treatment of Newtonian gravity with Newton's second law and the inverse-square law of gravitation:

$$m_I \boldsymbol{a} = \boldsymbol{F}\,, \tag{1.2a}$$

$$\boldsymbol{F} = -\frac{G m_G M}{r^2}\boldsymbol{n}\,. \tag{1.2b}$$

In the first equation, $\boldsymbol{F}$ is the force acting on a body of inertial mass m_I situated at position $\boldsymbol{r}(t)$, and $\boldsymbol{a} = d^2\boldsymbol{r}/dt^2$ is its acceleration. In the second equation, the force is assumed to be gravitational in nature, and to originate from a gravitating mass situated at the origin of the coordinate system. The force law involves m_G, the passive gravitational mass of the first body at $\boldsymbol{r}$, while M is the active gravitational mass of the second body. The quantity G is Newton's constant of gravitation, equal to $6.6738 \pm 0.0008 \times 10^{-11}\,\mathrm{m^2\,kg^{-1}\,s^{-2}}$. The force is attractive, it varies inversely with the square of the distance $r := |\boldsymbol{r}| = (x^2 + y^2 + z^2)^{1/2}$, and it points in the direction opposite to the unit vector $\boldsymbol{n} := \boldsymbol{r}/r$. An alternative form of the force law is obtained by writing it as the gradient of a potential $U = GM/r$, so that

$$\boldsymbol{F} = m_G \boldsymbol{\nabla} U\,. \tag{1.3}$$

This *Newtonian potential* will play a central role in virtually all chapters of this book.

If the inertial and passive gravitational masses of the body are equal to each other, $m_I = m_G$, then the acceleration of the body is given by $\boldsymbol{a} = \boldsymbol{\nabla} U$, and its magnitude is $a = GM/r^2$. Under this condition the acceleration is independent of the mass of the body. This statement is known as the *weak equivalence principle* (WEP), and it was a central element in Einstein's thinking on his way to the concepts of curved spacetime and general relativity. Although Newton did not explicitly use our formulation in terms of inertial and passive masses, he was well aware of the significance of their equality. In fact, he regarded this equality as so fundamental that he opened his treatise *Philosophiae Naturalis Principia Mathematica* with a discussion of it; he even alluded to his own experiments showing that the periods of pendulums were independent of the mass and type of material suspended, which establishes the equality of inertial and passive masses (he referred to them

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as the "quantity" and "weight" of bodies, respectively). Twentieth-century experiments have shown that the two types of mass are equal to parts in 10^{13} for a wide variety of materials (see Box 1.1).

Box 1.1 **Tests of the weak equivalence principle**

A useful way to discuss experimental tests of the weak equivalence principle is to parameterize the way it could be violated. In one parameterization, we imagine that a body is made up of atoms, and that the inertial mass m_I of an atom consists of the sum of all the mass and energy contributions of its constituents. But we suppose that the different forms of energy may contribute differently to the gravitational mass m_G than they do to m_I. One way to express this is to write

$$m_G = m_I(1 + \eta),$$

where η is a dimensionless parameter that measures the difference. Because different forms of energy arising from the relevant subatomic interactions (such as electromagnetic and nuclear interactions) contribute different amounts to the total, depending on atomic structure, η could depend on the type of atom. For example, electrostatic energy of the nuclear protons contributes a much larger fraction of the total mass for high-*Z* atoms than for low-*Z* atoms.

Using this parameterization, we find from Eq. (1.2) that the acceleration of the body is given by

$$\boldsymbol{a} = -\frac{m_G}{m_I}\frac{GM}{r^2}\boldsymbol{n} = -(1+\eta)\frac{GM}{r^2}\boldsymbol{n}.$$

The difference in acceleration between two materials of different composition will then be given by

$$\Delta\boldsymbol{a} = \boldsymbol{a}_1 - \boldsymbol{a}_2 = -(\eta_1 - \eta_2)\frac{GM}{r^2}\boldsymbol{n}.$$

One way to place a bound on $\eta_1 - \eta_2$ is to drop two different objects in the Earth's gravitational field ($g = GM/r^2 \approx 9.8\,\mathrm{m\,s^{-2}}$), and compare their accelerations, or how long they take to fall. Although legend has it that Galileo Galilei verified the equivalence principle by dropping objects off the Leaning Tower of Pisa around 1590, in fact experiments like this had already been performed and were well known to Galileo; if he did indeed drop things off the Tower, he may simply have been performing a kind of classroom demonstration of an established fact for his students. Unfortunately, the "Galileo approach" is plagued by experimental errors, such as the difficulty of releasing the objects at exactly the same time, by the effects of air drag, and by the short time available for timing the drop.

A better approach is to balance the gravitational force (which depends on m_G) by a support force (which depends on m_I); the classic model is the pendulum experiments performed by Newton and reported in his *Principia*. The period of the pendulum depends on m_G/m_I, g, and the length of the pendulum. These experiments are also troubled by air drag, by errors in measuring or controlling the length of the pendulum, and by errors in timing the swing.

The best approach for laboratory tests was pioneered by Baron Roland von Eötvös, a Hungarian geophysicist working around the turn of the 20th century. He developed the *torsion balance*, schematically consisting of a rod suspended by a wire near its mid-point, with objects consisting of different materials attached at each end.

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The point where the wire is attached to achieve a horizontal balance depends only on the gravitational masses of the two objects, so this configuration does not tell us anything. But if an additional gravitational force can be applied in a direction perpendicular to the supporting wire, and if there is a difference in m_G/m_I for the two bodies, then the rod will rotate in one direction or the other and the wire will twist until the restoring force of the twisted wire halts the rotation. There is no effect when m_G/m_I is the same for the two bodies. The additional force could be provided by a nearby massive body in the laboratory, a nearby mountain, the Sun, or the galaxy. Eötvös realized that, because of the centrifugal force produced by the rotation of the Earth, the wire hangs not exactly vertically, but is tilted slightly toward the south; at the latitude of Budapest, Hungary, the angle of tilt is about 0.1 degrees. Thus the gravitational acceleration of the Earth has a small component, about $g/400$, perpendicular to the wire, in a northerly direction. By slowly rotating the whole apparatus carefully about the vertical direction, Eötvös could compare the twist in two opposite orientations of the rod, and thereby eliminate a number of sources of error.

Eötvös found no measurable twist, within his experimental errors, for many different combinations of materials, and he was able to place an upper limit of $|\eta_1 - \eta_2| < 3 \times 10^{-9}$, corresponding to a limit on any difference in acceleration of the order of $7 \times 10^{-11}\ \mathrm{m\,s^{-2}}$. Even though the driving acceleration is only a tiny fraction of g, there is an enormous gain in sensitivity to tiny accelerations, mainly because the apparatus is almost static and can be observed for long periods of time. Torsion balance experiments were improved by Robert Dicke in Princeton and Vladimir Braginsky in Moscow during the 1960s and 1970s, and again during the 1980s as part of a search for a hypothetical "fifth" force (no evidence for such a force was found). The most recent experiments, performed notably by the "Eöt-Wash" group at the University of Washington, Seattle, have reached precisions of a few parts in 10^{13}; these experiments used the Sun or the galaxy as the source of gravity.

All these experiments exploit only a tiny fraction of the available acceleration. The only way to make full use of g while maintaining high sensitivity to acceleration differences is to design a "perpetual" Galileo drop experiment, namely by putting the different bodies in orbit around the Earth. Various satellite tests of the equivalence principle are in preparation, with the goal of reaching sensitivities ranging from 10^{-15} to 10^{-18}. Such experiments come with a high monetary cost: compared to laboratory experiments, space experiments are extraordinarily expensive.

Another test of the equivalence principle was carried out using the Earth–Moon system. The two bodies have slightly different compositions, with the Earth dominated by its iron–nickel core, and the Moon dominated by silicates. If there were a violation of the equivalence principle, the two bodies would fall with different accelerations toward the Sun, and this would have an effect on the Earth–Moon orbit. Lunar laser ranging is a technique of bouncing laser beams off reflectors placed on the lunar surface during the American and Soviet lunar landing programs of the 1970s, and it has reached the capability of measuring the Earth–Moon distance at the sub-centimeter level. No evidence for such a perturbation in the Earth–Moon distance has been found, so that the Earth and the Moon obey the equivalence principle to a few parts in 10^{13}. We describe the laser ranging measurements of the Moon in more detail in Box 13.2.

The weak equivalence principle is one of the most important foundational elements of relativistic theories of gravity. We will return to it in Chapter 5, on our way to general relativity.

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We shall assume that the weak equivalence principle holds perfectly, and make this an axiom of Newtonian gravity. We shall return to this principle in Chapter 5 and present it as an essential foundational element of general relativity, and we shall return to it again in Chapter 13 – in a different version known as the *strong equivalence principle* – and present it as a highly non-trivial property of massive, self-gravitating bodies in general relativity.

The weak equivalence principle allows us to rewrite Eqs. (1.2) in the form of an equation of motion for the body at $\boldsymbol{r}(t)$, and a field equation for the potential U:

$$\boldsymbol{a} = \boldsymbol{\nabla} U \,, \tag{1.4a}$$

$$U = GM/r \,. \tag{1.4b}$$

These equations are limited in scope, and they do not yet form the final set of equations that will be adopted as the foundations of Newtonian gravity. Their limitation has to do with the fact that they apply to a point mass situated at $\boldsymbol{r}(t)$ being subjected to the gravitational force produced by another point mass situated at the origin of the coordinate system. We are interested in much more general situations. First, we wish to consider the motion of extended bodies made up of continuous matter (solid, fluid, or gas), allowing the bodies to be of arbitrary size, shape, and constitution, and possibly to evolve in time according to their own internal dynamics. Second, we wish to consider an arbitrary number of such bodies, and to put them all on an equal footing; each body will be subjected to the gravity of the remaining bodies, and each will move in response to this interaction.

These goals can be achieved by generalizing the primitive Eqs. (1.4) to a form that applies to a continuous distribution of matter. We shall perform this generalization in Secs. 1.3 and 1.4, but to complete the discussion of this section, we choose to immediately list and describe the resulting equations.

Our formulation of the fundamental equations of Newtonian gravity relies on a fluid description of matter, in which the matter distribution is characterized by a mass-density field $\rho(t, \boldsymbol{x})$, a pressure field $p(t, \boldsymbol{x})$, and a velocity field $\boldsymbol{v}(t, \boldsymbol{x})$; these quantities depend on time t and position $\boldsymbol{x}$ within the fluid. Our formulation relies also on the Newtonian potential $U(t, \boldsymbol{x})$, which also depends on time and position, and which provides a description of the gravitational field. The equations that govern the behavior of the matter are the *continuity equation*,

$$\frac{\partial \rho}{\partial t} + \boldsymbol{\nabla} \cdot (\rho \boldsymbol{v}) = 0 \,, \tag{1.5}$$

which expresses the conservation of mass, and *Euler's equation*,

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{\nabla} U - \boldsymbol{\nabla} p \,, \tag{1.6}$$

which is the generalization of Eq. (1.4a) to continuous matter; here

$$\frac{d}{dt} := \frac{\partial}{\partial t} + \boldsymbol{v} \cdot \boldsymbol{\nabla} \,, \tag{1.7}$$

is the convective time derivative associated with the motion of fluid elements. The equation that governs the behavior of the gravitational field is *Poisson's equation*

$$\nabla^2 U = -4\pi G \rho \,, \tag{1.8}$$

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where

$$\nabla^2 := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \tag{1.9}$$

is the familiar Laplacian operator; Poisson's equation (known after its originator Siméon Denis Poisson, who unfortunately is not related to either author of this book), is the generalization of Eq. (1.4b) to continuous matter.

As was stated previously, these equations will be properly introduced in the following two sections. To complete the formulation of the theory we must impose a relationship between the pressure and the density of the fluid. This relationship, known as the *equation of state*, takes the general form of

$$p = p(\rho, T, \cdots), \tag{1.10}$$

in which the pressure is expressed as a function of the density, temperature, and possibly other relevant variables such as chemical composition. The equation of state encodes information about the microphysics that governs the fluid, and this information must be provided as an input in most applications of the theory.

A complete description of a physical situation involving gravity and a distribution of matter can be obtained by integrating Eqs. (1.5), (1.6), and (1.8) simultaneously and self-consistently. The solutions must be subjected to suitable boundary conditions, which will be part of the specification of the problem. All of Newtonian gravity is contained in these equations, and all associated phenomena follow as consequences of these equations.

1.3 Newtonian field equation

In this section we examine the equations that govern the behavior of the gravitational field, and show how Eq. (1.8) is an appropriate generalization of the more primitive form of Eq. (1.4b).

We recall that the relation $U = GM/r$ applies to a point body of active gravitational mass M situated at the origin of the coordinate system. Suppose that we are given an arbitrary number N of point bodies, and that we assign to each one a label $A = 1, 2, \cdots, N$. The mass and position of each body are then denoted M_A and $\boldsymbol{r}_A(t)$, respectively. If we *assume* that the total Newtonian potential U is a linear superposition of the individual potentials U_A created by each body, we have that the potential at position $\boldsymbol{x}$ is given by

$$U = \sum_A U_A = G \sum_A \frac{M_A}{|\boldsymbol{x} - \boldsymbol{r}_A|}. \tag{1.11}$$

The generalization of this relation to a continuous distribution of matter is straightforward. We convert the discrete sum $\sum_A M_A$ to a continuous integral $\int d^3x' \rho(t, \boldsymbol{x}')$, and we replace the discrete positions $\boldsymbol{r}_A$ with the continuous integration variable $\boldsymbol{x}'$. The result is

$$U(t, \boldsymbol{x}) = G \int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} d^3x', \tag{1.12}$$

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one of the key defining equations for the Newtonian potential. The integral can be evaluated as soon as the density field $\rho(t, \boldsymbol{x}')$ is specified, regardless of whether ρ is a proper solution to the remaining fluid equations. As such, Eq. (1.12) gives U as a *functional* of an arbitrary function ρ. The potential, however, will be physically meaningful only when ρ itself is physically meaningful, which means that it must be a proper solution to the continuity and Euler equations.

The integral equation (1.12) can easily be transformed into a differential equation for the Newtonian potential U. The transformation relies on the identity

$$\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = -4\pi\,\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{1.13}$$

in which $\delta(\boldsymbol{x} - \boldsymbol{x}') := \delta(x - x')\delta(y - y')\delta(z - z')$ is a three-dimensional delta function defined by the properties

$$\delta(\boldsymbol{x} - \boldsymbol{x}') = 0 \qquad \text{when } \boldsymbol{x} \neq \boldsymbol{x}', \tag{1.14a}$$

$$f(\boldsymbol{x})\delta(\boldsymbol{x} - \boldsymbol{x}') = f(\boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}') \qquad \text{for any smooth function } f(\boldsymbol{x}), \tag{1.14b}$$

$$\int \delta(\boldsymbol{x} - \boldsymbol{x}')\, d^3x' = 1 \qquad \text{for any domain of integration that encloses } \boldsymbol{x}. \tag{1.14c}$$

These properties further imply that $\delta(\boldsymbol{x}' - \boldsymbol{x}) = \delta(\boldsymbol{x} - \boldsymbol{x}')$. The identity of Eq. (1.13) is derived in Box 1.2. If we apply the Laplacian operator on both sides of Eq. (1.12) and exchange the operations of integration and differentiation on the right-hand side, we obtain

$$\begin{aligned}\nabla^2 U &= G \int \rho(t, \boldsymbol{x}')\nabla^2 \frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x' \\ &= -4\pi G \int \rho(t, \boldsymbol{x}')\delta(\boldsymbol{x} - \boldsymbol{x}')\, d^3x' \\ &= -4\pi G\rho(t, \boldsymbol{x});\end{aligned}$$

the identity was used in the second step, and the properties of the delta function displayed in Eq. (1.14) allowed us to evaluate the integral. The end result is Poisson's equation,

$$\nabla^2 U = -4\pi G\rho, \tag{1.15}$$

whose formulation was anticipated in Eq. (1.8).

It is possible to proceed in the opposite direction, and show that Eq. (1.12) provides a solution to Poisson's equation (1.15). A powerful tool in the integration of differential equations is the *Green's function* $G(\boldsymbol{x}, \boldsymbol{x}')$, a function of a field point $\boldsymbol{x}$ and a source point $\boldsymbol{x}'$. In the specific context of Poisson's equation, the Green's function is required to be a solution to

$$\nabla^2 G(\boldsymbol{x}, \boldsymbol{x}') = -4\pi\,\delta(\boldsymbol{x} - \boldsymbol{x}'), \tag{1.16}$$

which is recognized as a specific case of the general differential equation, corresponding to a point mass situated at $\boldsymbol{x}'$. Armed with such an object, a formal solution to Eq. (1.15) can be expressed as

$$U(t, \boldsymbol{x}) = G \int G(\boldsymbol{x}, \boldsymbol{x}')\rho(t, \boldsymbol{x}')\, d^3x'; \tag{1.17}$$

Box 1.2 Proof that $\nabla^2|\boldsymbol{x}-\boldsymbol{x}'|^{-1} = -4\pi\,\delta(\boldsymbol{x}-\boldsymbol{x}')$

To simplify the proof of Eq. (1.13) we set $\boldsymbol{x}' = \boldsymbol{0}$ without loss of generality; this can always be achieved by a translation of the coordinate system. This gives rise to the simpler equation

$$\nabla^2 r^{-1} = -4\pi\,\delta(\boldsymbol{x})\,, \tag{1}$$

with $r := |\boldsymbol{x}|$.

We first show that $\nabla^2 r^{-1} = 0$ whenever $\boldsymbol{x} \neq \boldsymbol{0}$. Derivatives of r^{-1} can be evaluated with the help of the identities

$$\frac{\partial r}{\partial x^j} = n_j\,, \qquad \frac{\partial n_j}{\partial x^k} = \frac{\partial n_k}{\partial x^j} = \frac{1}{r}\big(\delta_{jk} - n_j n_k\big)\,,$$

where $x^j := (x, y, z)$ is a component notation for the vector $\boldsymbol{x}$, $n^j := x^j/r$, and δ_{jk} is the Kronecker delta, equal to one when $j = k$ and zero otherwise. These equations hold provided that $r \neq 0$. According to this we have that

$$\frac{\partial}{\partial x^j} r^{-1} = -\frac{1}{r^2} n_j$$

and

$$\frac{\partial^2}{\partial x^j \partial x^k} r^{-1} = \frac{1}{r^3}\big(3 n_j n_k - \delta_{jk}\big)\,.$$

Because $\boldsymbol{n}$ is a unit vector, it follows that $\nabla^2 r^{-1} = 0$ whenever $r \neq 0$.

To handle the special case $r = 0$ we introduce the vector $\boldsymbol{j} := \boldsymbol{\nabla} r^{-1}$ and write the left-hand side of Eq. (1) as $\boldsymbol{\nabla}\cdot\boldsymbol{j}$. Integrating this over a volume V bounded by a spherical surface S of radius η, we obtain

$$\int_V \boldsymbol{\nabla}\cdot\boldsymbol{j}\, d^3x = \oint_S \boldsymbol{j}\cdot d\boldsymbol{S}$$

by virtue of Gauss's theorem. Here $d\boldsymbol{S}$ is an outward-directed surface element on S, which can be expressed as $d\boldsymbol{S} = \boldsymbol{n}\eta^2\, d\Omega$, with $d\Omega$ denoting an element of solid angle centered at $\boldsymbol{n}$. The vector $\boldsymbol{j}$ is equal to $-\eta^{-2}\boldsymbol{n}$ on S, and evaluating the surface integral returns -4π.

Because $\nabla^2 r^{-1}$ vanishes when $\boldsymbol{x} \neq 0$ and integrates to -4π whenever the integration domain encloses $\boldsymbol{x} = \boldsymbol{0}$, we conclude that it is distributionally equal to $-4\pi\,\delta(\boldsymbol{x})$. The proof is complete.

the steps involved in establishing that this U is indeed a solution to Poisson's equation are identical to those that previously led us to Eq. (1.15) from Eq. (1.12). The difference is that in the earlier derivation the identity of the Green's function was already known. In the approach described here, the result follows simply by virtue of Eq. (1.16). It is not difficult, of course, to identify the Green's function: comparison with Eq. (1.13) allows us to write

$$G(\boldsymbol{x}, \boldsymbol{x}') = \frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,. \tag{1.18}$$

Not surprisingly, the Green's function represents the potential of a point mass situated at $\boldsymbol{x}'$.

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1.4 Equations of hydrodynamics

In this section we develop the foundations for the equations of hydrodynamics, as displayed previously in Eqs. (1.5) and (1.6).

1.4.1 Motion of fluid elements

Definition of fluid element

We begin by describing any material body as being made up of *fluid elements*, volumes of matter that are very small compared to the size of the body, but very large compared to the inter-molecular distance, so that the element contains a macroscopic number of molecules. The fluid description of matter is a coarse-grained one in which the molecular fluctuations are smoothed over, and the fluid element is meant to represent a local average of the matter contained within. The coarse-graining could be described in great detail, for example, by introducing a microscopic density $\eta(t, \boldsymbol{x})$ that fluctuates wildly on the molecular scale, as well as smoothing function $w(|\boldsymbol{x} - \boldsymbol{x}'|)$ that varies over a much larger scale; the macroscopic density would then be defined as $\rho(t, \boldsymbol{x}) = \int \eta(t, \boldsymbol{x}')w(|\boldsymbol{x} - \boldsymbol{x}'|)\, d^3x'$. We will not go into such depth here, and keep the discussion at an intuitive, elementary level.

Each fluid element can be characterized by a mass density ρ (the mass of the element divided by its volume), a pressure p (the normal force per unit area acting on the surface of the element), and a velocity $\boldsymbol{v}$ (the average velocity of the molecules in the element). Other variables, such as viscosity, temperature, entropy, mean atomic weight, opacity, and so on, can also be introduced (some of these appear in Sec. 1.4.2). Apart from the velocity, all fluid variables are assumed to be measured by an observer who is momentarily at rest with respect to the fluid element. This description is adequate in a Newtonian setting, but it will have to be refined later, when we transition to the relativistic setting of Chapters 4 and 5.

Perhaps the most important aspect of a fluid element is that it keeps its contents intact as it moves within the fluid. During the motion the element may alter its shape and even its volume, but it will always contain the same collection of molecules; by definition no molecule is allowed to enter or leave the element. (It may be helpful to think of the molecules as being tagged, and of the fluid element as a bag that contains the tagged molecules.) A very important consequence of this property is that the total mass contained in a fluid element will never change; it is a constant of the element's motion.

Euler equation

We now apply Newton's laws to a selected fluid element of volume $\mathcal{V}$. The mass of the element is $\rho\mathcal{V}$, and from Newton's second law we have that

$$(\rho\mathcal{V})\boldsymbol{a} = \boldsymbol{F}\,, \tag{1.19}$$

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where $\boldsymbol{F}$ is the net force acting on the element, and $\boldsymbol{a}$ is its acceleration. This can be expressed as $d\boldsymbol{v}/dt$, the rate of change of the element's velocity vector as it moves within the fluid. It is important to observe that this rate of change follows the motion of the fluid element, and that it does not keep the spatial position fixed; this observation gives rise to an important distinction between the *convective*, or *Lagrangian, derivative* d/dt, which follows the motion of the fluid, and the *partial*, or *Eulerian, derivative* $\partial/\partial t$, which keeps the spatial position fixed.

The Lagrangian time derivative d/dt takes into account both the intrinsic time evolution of fluid variables and the variations that result from the motion of each fluid element. The fluid changes its configuration in a time interval dt, and a selected fluid element moves from an old position $\boldsymbol{x}$ to a new position $\boldsymbol{x} + d\boldsymbol{x}$. A fluid quantity $f(t, \boldsymbol{x})$, such as the mass density or a component of the velocity vector, changes by $df = f(t + dt, \boldsymbol{x} + d\boldsymbol{x}) - f(t, \boldsymbol{x})$ when we follow the motion of the fluid element. To first order in the displacement this is $df = (\partial f/\partial t)dt + (\boldsymbol{\nabla} f) \cdot d\boldsymbol{x}$, or

$$\begin{aligned} \frac{d}{dt} f(t, \boldsymbol{x}) &= \frac{\partial}{\partial t} f(t, \boldsymbol{x}) + \frac{d\boldsymbol{x}}{dt} \cdot \boldsymbol{\nabla} f(t, \boldsymbol{x}) \\ &= \frac{\partial}{\partial t} f(t, \boldsymbol{x}) + \boldsymbol{v} \cdot \boldsymbol{\nabla} f(t, \boldsymbol{x}) \,. \end{aligned} \tag{1.20}$$

This equation provides a link between the Lagrangian and Eulerian time derivatives.

Returning to Eq. (1.19), we assume that the force $\boldsymbol{F}$ acting on the fluid element comes from gravity and pressure gradients. By analogy with the expression in Eq. (1.3), the gravitational force is written as

$$\boldsymbol{F}_{\text{gravity}} = (\rho \mathcal{V}) \boldsymbol{\nabla} U \,, \tag{1.21}$$

where we assume that the inertial mass density and passive gravitational mass density are equal, as dictated by the weak equivalence principle. To derive an expression for the pressure-gradient force, we consider a cubic fluid element, and for the moment we focus our attention on the x-component of the force. The normal force acting on the face at $x = x_1$ is $p(x_1)\mathcal{A}$, in which $\mathcal{A}$ is the cross-sectional area of the fluid element. Similarly, the normal force acting on the face at $x = x_2 = x_1 + dx$ is $-p(x_2)\mathcal{A}$, with the minus sign accounting for the different directions of the normal vector. It follows that the net force acting in the x-direction is $(p_1 - p_2)\mathcal{A} \approx -(dp/dx)\Delta x \mathcal{A} = -(dp/dx)\mathcal{V}$. Generalizing to three dimensions, we find that the pressure-gradient force is given by

$$\boldsymbol{F}_{\text{pressure}} = -\mathcal{V} \boldsymbol{\nabla} p \,. \tag{1.22}$$

Inserting Eqs. (1.21) and (1.22) within Eq. (1.19) and dropping the common factor of $\mathcal{V}$, we obtain *Euler's equation* of hydrodynamics in a gravitational field,

$$\rho \frac{d\boldsymbol{v}}{dt} = \rho \boldsymbol{\nabla} U - \boldsymbol{\nabla} p \,. \tag{1.23}$$

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This equation (in spite of its name) is written in terms of the Lagrangian time derivative. An alternative formulation is

$$\rho\left[\frac{\partial \boldsymbol{v}}{\partial t} + (\boldsymbol{v}\cdot\boldsymbol{\nabla})\boldsymbol{v}\right] = \rho\boldsymbol{\nabla}U - \boldsymbol{\nabla}p\,, \tag{1.24}$$

and this involves the Eulerian time derivative.

Continuity equation

Conservation of the number of molecules in each fluid element implies that the mass of each element stays constant as it moves within the fluid. This is expressed mathematically as $d(\rho\mathcal{V})/dt = 0$, in terms of the Lagrangian time derivative. It is simple to show, however (see Box 1.3), that $\mathcal{V}^{-1}d\mathcal{V}/dt = \boldsymbol{\nabla}\cdot\boldsymbol{v}$, and the equation of mass conservation can be expressed as

$$\frac{d\rho}{dt} + \rho\boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\,. \tag{1.25}$$

The Eulerian form of this equation is

$$\frac{\partial\rho}{\partial t} + \boldsymbol{\nabla}\cdot(\rho\boldsymbol{v}) = 0\,, \tag{1.26}$$

and in this guise it is known as the *continuity equation*.

Box 1.3 **Proof that $\mathcal{V}^{-1}d\mathcal{V}/dt = \boldsymbol{\nabla}\cdot\boldsymbol{v}$**

Consider a cubic fluid element of sides L, volume $\mathcal{V} = L^3$, moving with an averaged velocity $\boldsymbol{v}$. The face of the cube at $x + L/2$ moves with a velocity $\boldsymbol{v}(x + L/2, y, z)$, while the face at $x - L/2$ moves with a velocity $\boldsymbol{v}(x - L/2, y, z)$. In a time Δt the length of the cube in the *x*-direction changes by $[v_x(x + L/2, y, z) - v_x(x - L/2, y, z)]\Delta t \approx L(dv_x/dx)\Delta t$. Repeating this argument for the *y* and *z*-directions, we find that the change in the cube's volume is

$$\Delta\mathcal{V} \approx L^3\left(1 + \frac{dv_x}{dx}\Delta t\right)\left(1 + \frac{dv_y}{dy}\Delta t\right)\left(1 + \frac{dv_z}{dz}\Delta t\right) - L^3 \approx \mathcal{V}\boldsymbol{\nabla}\cdot\boldsymbol{v}\Delta t\,.$$

Taking the limit $\Delta t \to 0$, we obtain the desired result.

1.4.2 Thermodynamics of fluid elements

We now focus our attention on a selected fluid element. We assume that the molecular mean free path (the average distance travelled by a molecule between collisions) as well as the photon mean free path (the average distance travelled by a photon before being scattered or absorbed by a molecule) are both very small compared to the size of the fluid element. Equivalently, we assume that the time required for the fluid element to change in a significant way is very long compared to the time scales that characterize interactions among molecules and photons within the fluid element. It follows from these assumptions that at any given

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moment of time, the fluid element can achieve a state of *local thermodynamic equilibrium* in which its matter content is characterized by a local temperature $T(t, \boldsymbol{x})$, and in which the photons are characterized by a black-body spectrum at the same temperature. We can also ascribe a local entropy $\mathcal{S}(t, \boldsymbol{x})$, a local internal energy $\mathcal{E}(t, \boldsymbol{x})$, and other thermodynamic variables to the fluid element. These quantities may vary from one fluid element to the next, and they may vary with time, but they do so on time and distance scales that are long compared to those associated with the molecular processes that drive each element toward equilibrium. We can therefore apply the laws of thermodynamics locally to each fluid element.

First and second laws

The first law of thermodynamics, applied to a fluid element, reads

$$d\mathcal{E} = \delta Q + \delta \mathcal{W}, \tag{1.27}$$

in which $\mathcal{E}$ is the internal energy of the fluid element, $\delta\mathcal{W} = -p\, d\mathcal{V}$ the work done on the fluid element, and δQ the heat absorbed. This can be expressed as

$$\delta Q = (\rho\mathcal{V})q\, dt - \mathcal{V}\nabla \cdot \boldsymbol{H}\, dt, \tag{1.28}$$

where q is the energy per unit mass generated within the fluid element per unit time, and $\boldsymbol{H}$ is the heat-flux vector, defined in such a way that $\boldsymbol{H} \cdot d\boldsymbol{S}$ is the heat crossing an element of surface area (described by $d\boldsymbol{S}$) per unit time. In general, ρq represents heat that is generated internally (for example by chemical or nuclear reactions), and $\nabla \cdot \boldsymbol{H}$ represents heat imported from neighboring fluid elements (for example by heat conduction or radiation). The second term can be motivated by considering a cubic element, and examining the heat entering the element from the x-direction. The heat absorbed in a time dt is given by the flux $H_x(x)$ entering the face at x times the area $\mathcal{A}$ of that face, minus the flux $H_x(x + dx)$ leaving at $x + dx$ times the area $\mathcal{A}$. The net result is $-\partial H_x/dx(\mathcal{A}\, dx)$, and including the y and z directions gives $-\mathcal{V}\nabla \cdot \boldsymbol{H}\, dt$, as required.

Defining the energy density $\epsilon := \mathcal{E}/\mathcal{V}$, we can rewrite the first law of thermodynamics in the form

$$d\epsilon - \frac{\epsilon + p}{\rho} d\rho = (\rho q - \nabla \cdot \boldsymbol{H})\, dt. \tag{1.29}$$

A useful alternative variable is the internal energy per unit mass $\Pi := \epsilon/\rho$, for which the first law takes the form

$$d\Pi + p\, d\left(\frac{1}{\rho}\right) = \left(q - \frac{1}{\rho}\nabla \cdot \boldsymbol{H}\right) dt. \tag{1.30}$$

The second law of thermodynamics states that for any reversible process, $\delta Q = T\, d\mathcal{S}$, where $\mathcal{S}$ is the entropy of the fluid element. Introducing the entropy per unit mass $s := \mathcal{S}/(\rho\mathcal{V})$, we have that

$$T\, ds = \left(q - \frac{1}{\rho}\nabla \cdot \boldsymbol{H}\right) dt, \tag{1.31}$$

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and the first law can now be expressed as

$$d\Pi + p\, d\left(\frac{1}{\rho}\right) = T\, ds\,. \tag{1.32}$$

As a special case of these laws, we consider a situation in which the fluid element, in addition to being in local thermodynamic equilibrium, is also in *thermal equilibrium* with neighboring elements. In such circumstances there is no net transfer of heat, and the element evolves adiabatically, with $ds = 0$. This requires that

$$\boldsymbol{\nabla} \cdot \boldsymbol{H} = \rho q\,, \tag{1.33}$$

and the first law can then be expressed in the restricted form

$$d\Pi = -p\, d\left(\frac{1}{\rho}\right) = \frac{p}{\rho^2}\, d\rho\,. \tag{1.34}$$

Equation of state

Given a system of known composition, labelled symbolically by X, there exists a relation $p = p(\rho, T; X)$ between the pressure, density, and temperature called the *equation of state*. The equation of state is a necessary input into any application of the laws of thermodynamics, and to complete our discussion we provide a brief review of some of the equations of state that are relevant to the description of stellar configurations. We make no attempt to be complete here, as equations of state are the subject of a multitude of textbooks on statistical mechanics and thermodynamics.

The temperature inside most main-sequence stars is extremely high, and typically the kinetic energy of the atoms is very large compared to their interaction energy; the stellar matter can therefore be taken to be non-interacting, and to make up an ideal gas. Most stellar interiors are completely ionized, and the free electrons can also be treated as an ideal gas. The equation of state is then the familiar $p = nkT$, where n is the number density and k is Boltzmann's constant. The total pressure is the sum of the partial pressures, and the ionic contribution is

$$p_I = n_I kT = \frac{\rho}{\mu_I m_{\rm H}} kT\,, \tag{1.35}$$

where $m_{\rm H}$ is the atomic mass unit, and μ_I is the mean atomic number of the ions. This is defined by

$$\frac{1}{\mu_I} := \sum_i \frac{X_i}{\mathcal{A}_i}\,, \tag{1.36}$$

where X_i is the fraction by mass of the ith species ($\sum_i X_i = 1$), and $\mathcal{A}_i$ is its atomic mass number. For stars in which hydrogen and helium dominate over heavier elements (called *metals* by stellar astrophysicists), one often writes $\mu_I^{-1} := X + \frac{1}{4}Y + (1 - X - Y)\langle \mathcal{A}^{-1}\rangle$, in which X is the mass fraction of hydrogen, Y is the mass fraction of helium, and $\langle \mathcal{A}^{-1}\rangle$ is an average of $\mathcal{A}_i^{-1}$ over the metals.

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For the electrons we have that

$$p_e = n_e kT = \frac{\rho}{\mu_e m_{\rm H}} kT\,, \tag{1.37}$$

where

$$\frac{1}{\mu_e} := \sum_i \frac{Z_i X_i}{\mathcal{A}_i}, \tag{1.38}$$

with Z_i denoting the atomic number of the ith ionic species. Because $Z_i/\mathcal{A}_i \approx 1/2$ for most elements except hydrogen (for which $Z/A = 1$), we can approximate μ_e^{-1} by $X + \frac{1}{2}(1 - X) = \frac{1}{2}(1 + X)$ for most stellar materials. The total gas pressure is then

$$p_{\rm gas} = \left(\frac{1}{\mu_I} + \frac{1}{\mu_e}\right) \frac{\rho}{m_{\rm H}} kT := \frac{\rho}{\mu m_{\rm H}} kT\,, \tag{1.39}$$

where $\mu^{-1} := \mu_I^{-1} + \mu_e^{-1}$. The energy density of such a classical ideal gas is given by $\epsilon_{\rm gas} = \frac{3}{2} p_{\rm gas}$.

Another important constituent of stars is radiation. As we have seen, under conditions of local thermodynamic equilibrium (which are upheld in stellar interiors) the radiation within each fluid element can be treated as a black body of the same temperature T as the fluid element. The equation of state and energy density for the radiation are given by

$$p_{\rm rad} = \frac{1}{3} a T^4\,, \tag{1.40}$$

and $\epsilon_{\rm rad} = aT^4 = 3p_{\rm rad}$, where

$$a := \frac{8\pi^5 k^4}{15 h^3 c^3} \tag{1.41}$$

is the radiation constant ($\sigma := \frac{1}{4}ac$ is the Stefan-Boltzmann constant). The total pressure inside a star is then $p = p_{\rm gas} + p_{\rm rad}$, and the total energy density is $\epsilon = \epsilon_{\rm gas} + \epsilon_{\rm rad}$.

At the sufficiently high densities that characterize dead stars such as white dwarfs and neutron stars, matter becomes *degenerate*, and the equation of state changes dramatically. This occurs when the temperature T and number density n are such that the characteristic momentum (or uncertainty in the momentum) of a particle of mass m, $\Delta p \sim \sqrt{mkT}$, multiplied by the typical interparticle distance $\Delta x \sim n^{-1/3}$, starts running afoul of the Heisenberg uncertainty principle, which requires that $\Delta x \Delta p \geq \hbar$. This state of degeneracy occurs when $mkT \leq \hbar^2 n^{2/3}$, or when

$$T \leq T_{\rm F}\,, \qquad T_{\rm F} := \frac{\hbar^2}{2km}(3\pi^2 n)^{2/3} \tag{1.42}$$

after inserting the appropriate numerical coefficients. Here $T_{\rm F}$ is the *Fermi temperature* associated with a free fermion gas of number density n and constituent mass m. For a white dwarf, the electrons are degenerate, while the ions, being at least 2000 times more massive, are not. In a neutron star, as a consequence of the much higher density, the neutrons and the residual protons and electrons are degenerate. In laboratory situations involving low densities, the Fermi temperature is typically extremely low, and normal matter is rarely degenerate (an exception is the conduction electrons in metals, for which the Fermi

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temperature is much higher than room temperature). In a white dwarf, by contrast, the high densities involved imply that the Fermi temperature is of the order of 10^9 K, while the star's actual temperature typically ranges between 10^6 K and 10^7 K. For the even higher densities associated with neutron stars, the ratio T/T_F is even smaller; in this case the Fermi temperature is of order 10^{12} K, while the star's actual temperature is also comparable to 10^6 K.

To conclude our discussion we review the important case of the *polytropic* equation of state, in which p is related directly to the density, and in which T has been eliminated by assuming that each fluid element is in thermal equilibrium with neighboring elements. Under these conditions Eqs. (1.29) and (1.33) imply that $\rho\, d\epsilon - (\epsilon + p)\, d\rho = 0$, and we further assume that the fluid is such that the energy density is proportional to the pressure, so that

$$\epsilon = \eta\, p\,. \tag{1.43}$$

The dimensionless constant η (usually denoted n, which is avoided here because n has already been assigned the meaning of number density) is known as the *polytropic index*; we have seen that $\eta = \frac{3}{2}$ for an ideal gas, while $\eta = 3$ for a photon gas. Combining these relations we find that $\eta\rho\, dp - (\eta + 1) p\, d\rho = 0$, and this can be integrated to yield

$$p = K\rho^{\Gamma}\,, \qquad \Gamma := 1 + 1/\eta\,, \tag{1.44}$$

where K is an integration constant. This is the polytropic equation of state, which relates pressure and density during an adiabatic thermodynamic process; the exponent Γ is known as the *adiabatic index*.

1.4.3 Global conservation laws

The equations of hydrodynamics give rise to a number of important *global conservation laws*. These refer to global quantities, defined as integrals over the entire fluid system, that are constant in time whenever the system is *isolated*, that is, whenever the system is not affected by forces external to it. For fluids subjected to pressure forces and Newtonian gravity, the globally conserved quantities are total mass, momentum, energy, and angular momentum. Because these are fundamentally important in any physical context, we examine them in detail here, providing precise definitions and proofs of their conservation. For these derivations we introduce a number of mathematical tools that will prove helpful throughout this book.

Integral identities

The conserved quantities are all defined as integrals over a volume of space that contains the entire isolated system. The domain of integration V is largely arbitrary, and is constrained by only two essential conditions: it must be a fixed region of space that does not evolve in time, and it must contain all the matter. It is useful to think of this domain as extending beyond the matter; it could, in fact, extend all the way to infinity. An essential property of

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the boundary S of the region of integration is that all matter variables (such as the mass density ρ and the pressure p) vanish on S.

The global quantities are integrals of the form $\int_V f(t,\boldsymbol{x})\,d^3x$, in which $f(t,\boldsymbol{x})$ is a function of time and space that will typically involve the fluid variables. The integral itself is a function of time only; to simplify the notation we shall henceforth omit the label V on the integration symbol. For any such integral we have that

$$\frac{d}{dt}\int f(t,\boldsymbol{x})\,d^3x = \int \frac{\partial f}{\partial t}\,d^3x\,. \tag{1.45}$$

This property follows because V is independent of time, and because the variable of integration $\boldsymbol{x}$ also is independent of time.

We next consider integrals of the form $F(t) := \int \rho(t,\boldsymbol{x}) f(t,\boldsymbol{x})\,d^3x$, in which a factor of the mass density ρ was extracted from the original function f. As we shall prove below, such integrals obey the identity

$$\frac{d}{dt}\int \rho(t,\boldsymbol{x}) f(t,\boldsymbol{x})\,d^3x = \int \rho \frac{df}{dt}\,d^3x\,, \tag{1.46}$$

in which, as usual,

$$\frac{df}{dt} = \frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f \tag{1.47}$$

is the convective (or Lagrangian) time derivative.

We may generalize the result by allowing f to depend on two position vectors, $\boldsymbol{x}$ and $\boldsymbol{x}'$. We define the integral $F(t,\boldsymbol{x}) := \int \rho(t,\boldsymbol{x}') f(t,\boldsymbol{x},\boldsymbol{x}')\,d^3x'$ and apply Eq. (1.46) to it. Because F depends on $\boldsymbol{x}$ in addition to t, the time derivative is correctly interpreted as a partial derivative that keeps the spatial variables fixed, and we obtain

$$\frac{\partial F}{\partial t} = \int \rho' \left(\frac{\partial f}{\partial t} + \boldsymbol{v}'\cdot\nabla' f\right) d^3x'\,, \tag{1.48}$$

in which ρ' is the mass density expressed as a function of t and $\boldsymbol{x}'$, $\boldsymbol{v}'$ is the velocity field expressed in terms of the same variables, and ∇' is the gradient operator associated with $\boldsymbol{x}'$. Now, the *Lagrangian time derivative* acting on F is $dF/dt = \partial F/\partial t + \boldsymbol{v}\cdot\nabla F$, and from Eq. (1.48) and the definition of $F(t,\boldsymbol{x})$ we find that this can be expressed as

$$\frac{d}{dt}\int \rho(t,\boldsymbol{x}') f(t,\boldsymbol{x},\boldsymbol{x}')\,d^3x' = \int \rho' \frac{df}{dt}\,d^3x'\,, \tag{1.49}$$

with

$$\frac{df}{dt} := \frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\nabla f + \boldsymbol{v}'\cdot\nabla' f \tag{1.50}$$

denoting a generalized Lagrangian derivative.

For a final application of Eq. (1.46) we define $\mathcal{F}(t) := \int \rho(t,\boldsymbol{x}) F(t,\boldsymbol{x})\,d^3x = \int \rho\rho' f(t,\boldsymbol{x},\boldsymbol{x}')\,d^3x' d^3x$. According to Eqs. (1.46) and (1.49) we find that the time derivative of this integral is given by

$$\frac{d}{dt}\int \rho(t,\boldsymbol{x})\rho(t,\boldsymbol{x}') f(t,\boldsymbol{x},\boldsymbol{x}')\,d^3x' d^3x = \int \rho\rho' \frac{df}{dt}\,d^3x' d^3x\,, \tag{1.51}$$

in which df/dt is once more given by Eq. (1.50).

We have yet to establish Eq. (1.46). The steps are straightforward, and they rely on the continuity equation (1.26), Gauss's theorem, and the fact that ρ vanishes on the boundary S of the domain of integration. We have

$$\begin{aligned}\frac{d}{dt}\int \rho(t,\boldsymbol{x}) f(t,\boldsymbol{x})\, d^3x &= \int\left(\rho\frac{\partial f}{\partial t} + f\frac{\partial\rho}{\partial t}\right) d^3x \\ &= \int\left(\rho\frac{\partial f}{\partial t} - f\boldsymbol{\nabla}\cdot(\rho\boldsymbol{v})\right) d^3x \\ &= \int\left(\rho\frac{\partial f}{\partial t} + \rho\boldsymbol{v}\cdot\boldsymbol{\nabla} f\right) d^3x - \oint f\rho\boldsymbol{v}\cdot d\boldsymbol{S} \\ &= \int \rho\frac{df}{dt}\, d^3x\,. \end{aligned} \tag{1.52}$$

The continuity equation was used in the second step. In the third step the volume integral of the total divergence $\boldsymbol{\nabla}\cdot(f\rho\boldsymbol{v})$ was expressed as a surface integral, which vanishes because $\rho = 0$ on S. In the fourth step we recover Eq. (1.46), as required.

Mass, momentum, and center-of-mass

The *total mass* of the fluid system is

$$M := \int \rho(t,\boldsymbol{x})\, d^3x\,. \tag{1.53}$$

While the integral should in principle be a function of time, it is a direct consequence of Eq. (1.46) – applied with $f = 1$ – that $dM/dt = 0$. The total mass of the fluid system is a conserved quantity that does not change with time. This is an obvious consequence of the fact that mass is conserved within each fluid element.

The *total momentum* of the fluid system is

$$\boldsymbol{P} := \int \rho(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})\, d^3x\,. \tag{1.54}$$

To verify that this is also a conserved quantity, we apply Eq. (1.46) with $f = \boldsymbol{v}$ and get

$$\frac{d\boldsymbol{P}}{dt} = \int \rho\frac{d\boldsymbol{v}}{dt}\, d^3x = \int \rho\boldsymbol{\nabla} U\, d^3x - \int \boldsymbol{\nabla} p\, d^3x \tag{1.55}$$

after inserting Euler's equation (1.23). The pressure integral is easy to dispose of: applying Gauss's theorem we find that it is equal to $\oint p\, d\boldsymbol{S}$, and this vanishes because $p = 0$ everywhere on S. The integral involving the Newtonian potential requires more work, but we shall show presently that

$$\int \rho\boldsymbol{\nabla} U\, d^3x = 0\,, \tag{1.56}$$

a result that is fundamentally important in the Newtonian theory of gravity. With all this we find that $d\boldsymbol{P}/dt = 0$, and conclude that total momentum is indeed conserved. This is a consequence of (or a statement of) Newton's third law, the equality of action and reaction.

The *center-of-mass* of the fluid system is situated at a position $\boldsymbol{R}(t)$ defined by

$$\boldsymbol{R}(t) := \frac{1}{M}\int \rho(t,\boldsymbol{x})\boldsymbol{x}\, d^3x\,. \tag{1.57}$$

Because M is conserved, the center-of-mass velocity $\boldsymbol{V} := d\boldsymbol{R}/dt$ is given by

$$\boldsymbol{V} := \frac{1}{M}\int \rho(t,\boldsymbol{x})\boldsymbol{v}(t,\boldsymbol{x})\, d^3x\,, \tag{1.58}$$

as obtained from Eq. (1.46) by applying the identity $dx^j/dt = \partial x^j/\partial t + \boldsymbol{v}\cdot\boldsymbol{\nabla}x^j = v^j$ to each component of $\boldsymbol{x}$. The integral is recognized as the total momentum, and we find that $\boldsymbol{V}$ is a conserved quantity. It follows that the center-of-mass moves according to

$$\boldsymbol{R}(t) = \boldsymbol{R}(0) + \boldsymbol{V}t\,, \tag{1.59}$$

with $\boldsymbol{V} := \boldsymbol{P}/M$. It is always possible to choose a reference frame such that $\boldsymbol{R}(0) = \boldsymbol{0}$ and $\boldsymbol{V} = \boldsymbol{0}$, so that $\boldsymbol{R}(t) = \boldsymbol{0}$; this defines the *center-of-mass frame* of the fluid system.

To establish Eq. (1.56) we recall the expression of Eq. (1.12) for the gravitational potential, on which we apply the gradient operator. Focusing our attention on the x^j component of $\boldsymbol{\nabla}U$, we have that

$$\frac{\partial U}{\partial x^j} = G\int \rho' \frac{\partial}{\partial x^j}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'. \tag{1.60}$$

The partial derivative can be evaluated explicitly, and can be seen to be equal to $-|\boldsymbol{x}-\boldsymbol{x}'|^{-3}(x^j - x'^j)$. Inserting $\partial U/\partial x^j$ within the integral of Eq. (1.56), we find that

$$\int \rho\frac{\partial U}{\partial x^j}\, d^3x = G\int \rho\rho' \frac{\partial}{\partial x^j}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' d^3x\,. \tag{1.61}$$

To show that this vanishes we employ a clever trick that will recur frequently throughout this book. It consists of swapping the variables of integration ($\boldsymbol{x} \leftrightarrow \boldsymbol{x}'$), and of writing the integral in the alternative form

$$\begin{aligned}\int \rho\frac{\partial U}{\partial x^j}\, d^3x &= G\int \rho'\rho \frac{\partial}{\partial x'^j}\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d^3x d^3x' \\ &= G\int \rho\rho' \frac{\partial}{\partial x'^j}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' d^3x\,.\end{aligned} \tag{1.62}$$

Explicitly we find that the partial derivative with respect to x'^j is equal to $+|\boldsymbol{x}-\boldsymbol{x}'|^{-3}(x^j - x'^j)$, which is equal and opposite to the derivative with respect to x^j. This property follows directly from the fact that $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ depends on the *difference* between $\boldsymbol{x}$ and $\boldsymbol{x}'$. Taking this property into account in Eq. (1.62), we find that

$$\int \rho\frac{\partial U}{\partial x^j}\, d^3x = -G\int \rho\rho' \frac{\partial}{\partial x^j}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' d^3x\,. \tag{1.63}$$

Comparing Eqs. (1.61) and (1.63), we find that the integral vanishes, as was already stated in Eq. (1.56).

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Energy

The *total energy* of a fluid system comprises three components. The first is the kinetic energy

$$\mathcal{T}(t) := \frac{1}{2}\int \rho v^2\, d^3x\,, \tag{1.64}$$

the second is the gravitational potential energy

$$\Omega(t) := -\frac{1}{2}\int \rho U\, d^3x = -\frac{1}{2}G\int \frac{\rho\rho'}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'd^3x\,, \tag{1.65}$$

and the third is the internal thermodynamic energy

$$E_{\rm int}(t) := \int \epsilon\, d^3x = \int \rho\Pi\, d^3x\,. \tag{1.66}$$

In these expressions, ρ is the mass density expressed as a function of t and $\boldsymbol{x}$, ρ' is the mass density expressed in terms of t and $\boldsymbol{x}'$, $v^2 := \boldsymbol{v}\cdot\boldsymbol{v}$ is the square of the velocity vector, ϵ is the density of internal thermodynamic energy, and $\Pi := \epsilon/\rho$ is the specific internal energy. The total energy is

$$E := \mathcal{T}(t) + \Omega(t) + E_{\rm int}(t)\,, \tag{1.67}$$

and while $\mathcal{T}$, Ω, and $E_{\rm int}$ can each vary with time, we shall prove that E is a conserved quantity. The definition provided here for total kinetic energy is immediately plausible: we take the kinetic energy of each fluid element, $\frac{1}{2}(\rho\mathcal{V})v^2$, and integrate over the entire fluid. The definition of total internal energy is also immediately plausible. The definition of total gravitational potential energy is more subtle, and its suitability is ultimately justified by the fact that the total energy turns out to be conserved. Nevertheless, we may observe that Ω is $-(\rho\mathcal{V})U$, the potential energy of each fluid element in the field of all other elements, integrated over the entire fluid; the factor of $\frac{1}{2}$ is inserted to avoid a double counting of pairs of fluid elements.

To prove that E is conserved we calculate how each term in Eq. (1.67) changes with time. We begin with $\mathcal{T}$, and get

$$\frac{d\mathcal{T}}{dt} = \int \rho\boldsymbol{v}\cdot\frac{d\boldsymbol{v}}{dt}\, d^3x = \int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\, d^3x - \int \boldsymbol{v}\cdot\boldsymbol{\nabla}p\, d^3x \tag{1.68}$$

after involving Euler's equation. The first integral can be expressed as

$$\int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\, d^3x = G\int \rho\rho'\,\boldsymbol{v}\cdot\boldsymbol{\nabla}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'd^3x\,, \tag{1.69}$$

or it can be expressed as

$$\int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\, d^3x = G\int \rho'\rho\,\boldsymbol{v}'\cdot\boldsymbol{\nabla}'\frac{1}{|\boldsymbol{x}'-\boldsymbol{x}|}\, d^3x d^3x' \tag{1.70}$$

by exploiting the "switch trick" introduced after Eq. (1.61). Adding the two expressions and dividing by 2, we obtain

$$\int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\, d^3x = \frac{1}{2}G\int \rho\rho'\big(\boldsymbol{v}\cdot\boldsymbol{\nabla} + \boldsymbol{v}'\cdot\boldsymbol{\nabla}'\big)\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'd^3x\,. \tag{1.71}$$

Because $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ does not depend on time, we may re-express this as

$$\int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\,d^3x = \frac{1}{2}G\int \rho\rho'\frac{d}{dt}\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|}\,d^3x'd^3x\,, \tag{1.72}$$

in which d/dt is the generalized Lagrangian derivative of Eq. (1.50). Invoking now the integral identity of Eq. (1.51), applied with $f=|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$, as well as the definition of Ω provided in Eq. (1.65), we finally arrive at

$$\int \rho\boldsymbol{v}\cdot\boldsymbol{\nabla}U\,d^3x = -\frac{d\Omega}{dt}\,, \tag{1.73}$$

which tells us how Ω changes with time.

Returning to Eq. (1.68), we next examine the second integral, in which we express $\boldsymbol{v}\cdot\boldsymbol{\nabla}p$ as $\boldsymbol{\nabla}\cdot(p\boldsymbol{v})-p\boldsymbol{\nabla}\cdot\boldsymbol{v}$. The total divergence gives no contribution (because p vanishes on S), and Eq. (1.25) implies that $\boldsymbol{\nabla}\cdot\boldsymbol{v}=-\rho^{-1}d\rho/dt$. All this gives us

$$\int \boldsymbol{v}\cdot\boldsymbol{\nabla}p\,d^3x = \int \frac{p}{\rho}\frac{d\rho}{dt}\,d^3x \tag{1.74}$$

for the second integral. Inserting this and Eq. (1.73) within Eq. (1.68), we finally obtain

$$\frac{d\mathcal{T}}{dt} = -\frac{d\Omega}{dt} - \int \frac{p}{\rho}\frac{d\rho}{dt}\,d^3x \tag{1.75}$$

for the rate of change of the total kinetic energy.

The final step is to compute $dE_{\rm int}/dt$. Starting with Eq. (1.66) and involving Eq. (1.46) with $f=\Pi$, we find that

$$\frac{dE_{\rm int}}{dt} = \int \rho\frac{d\Pi}{dt}\,d^3x\,. \tag{1.76}$$

Assuming that each fluid element is at all times in thermal equilibrium with neighboring elements, we invoke the first law of thermodynamics as stated in Eq. (1.34): $\rho\,d\Pi=(p/\rho)\,d\rho$. This gives

$$\frac{dE_{\rm int}}{dt} = \int \frac{p}{\rho}\frac{d\rho}{dt}\,d^3x \tag{1.77}$$

for the rate of change of the total internal energy. Combining Eqs. (1.73), (1.75), and (1.77), we find that $dE/dt=0$, and arrive at the conclusion that E is indeed conserved.

Angular momentum

The *total angular momentum* of a fluid system is defined by

$$\boldsymbol{J} := \int \rho\boldsymbol{x}\times\boldsymbol{v}\,d^3x\,. \tag{1.78}$$

The steps required to show that the angular momentum is conserved are now familiar. We use Eq. (1.46) with $f=\boldsymbol{x}\times\boldsymbol{v}$ to evaluate $d\boldsymbol{J}/dt$, and obtain

$$\frac{d\boldsymbol{J}}{dt} = \int \rho\boldsymbol{x}\times\boldsymbol{\nabla}U\,d^3x - \int \boldsymbol{x}\times\boldsymbol{\nabla}p\,d^3x \tag{1.79}$$

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after inserting Euler's equation. The first integral is evaluated as

$$\begin{aligned}\int \rho \boldsymbol{x} \times \boldsymbol{\nabla} U \, d^3x &= -G \int \rho\rho' \frac{\boldsymbol{x} \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d^3x' d^3x \\ &= G \int \rho\rho' \frac{\boldsymbol{x}' \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d^3x' d^3x \\ &= -\frac{1}{2} G \int \rho\rho' \frac{(\boldsymbol{x} - \boldsymbol{x}') \times (\boldsymbol{x} - \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d^3x' d^3x \,,\end{aligned}$$

so that

$$\int \rho \boldsymbol{x} \times \boldsymbol{\nabla} U \, d^3x = 0 \,. \tag{1.80}$$

The "switch trick" was exploited in the second step, and in the third step we added the two expressions and divided by 2. For the second integral we make use of the vector-algebra identity $\boldsymbol{x} \times \boldsymbol{\nabla} p = -\boldsymbol{\nabla} \times (p\boldsymbol{x}) + p\boldsymbol{\nabla} \times \boldsymbol{x}$; the second term vanishes identically, and the integral of the first term can be expressed as a vanishing surface integral. Thus

$$\int \boldsymbol{x} \times \boldsymbol{\nabla} p \, d^3x = 0 \,, \tag{1.81}$$

and we have arrived at the conservation statement $d\boldsymbol{J}/dt = 0$.

Virial theorems

Another important set of global relations satisfied by an isolated fluid system is known as the *virial theorems*. They involve a number of new global quantities. The first is

$$I^{jk}(t) := \int \rho(t, \boldsymbol{x}) x^j x^k \, d^3x \,, \tag{1.82}$$

the *quadrupole moment tensor* of the mass distribution, an object that will accompany us throughout this book. The second is

$$\mathcal{T}^{jk}(t) := \frac{1}{2} \int \rho v^j v^k \, d^3x \,, \tag{1.83}$$

the *kinetic energy tensor* of the fluid system, a tensorial generalization of $\mathcal{T}$ defined by Eq. (1.64); it is easy to see that $\mathcal{T}$ is the trace of the kinetic energy tensor. The third is

$$\Omega^{jk}(t) := -\frac{1}{2} G \int \rho\rho' \frac{(x - x')^j (x - x')^k}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d^3x' d^3x \,, \tag{1.84}$$

the *gravitational energy tensor* of the fluid system, a tensorial generalization of Ω defined by Eq. (1.65); once again it is easy to see that Ω is the trace of the gravitational energy tensor. And finally, the virial theorems involve

$$P(t) := \int p \, d^3x \,, \tag{1.85}$$

the *integrated pressure* of the fluid system.

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The tensorial version of the virial theorem is a statement about the second time derivative of the quadrupole moment tensor. Applying these derivatives to Eq. (1.82) and making use of Eq. (1.46) yields

$$\frac{d^2 I^{jk}}{dt^2} = 2\int \rho v^j v^k \, d^3x + 2\int \rho x^{(j} \frac{dv^{k)}}{dt} \, d^3x \,, \tag{1.86}$$

where we inserted parentheses around the indices in the second integral to indicate symmetrization: $x^{(j} dv^{k)}/dt := \frac{1}{2}(x^j dv^k/dt + x^k dv^j/dt)$. (The operations of symmetrization and antisymmetrization of tensorial indices are described more fully in Box 1.4). Inserting now the Euler equation within Eq. (1.86), we obtain

$$\begin{aligned}\frac{d^2 I^{jk}}{dt^2} = {} & 2\int \rho v^j v^k \, d^3x - 2G \int \rho\rho' \frac{x^{(j}(x - x')^{k)}}{|\boldsymbol{x} - \boldsymbol{x}'|^3} \, d^3x' d^3x \\ & - 2\int x^{(j}\partial^{k)} p \, d^3x \,,\end{aligned} \tag{1.87}$$

in which $\partial^k p$ is a shorthand notation for $\partial p/\partial x^k$. To proceed we exploit the "switch trick" in the second integral, and integrate the third integral by parts. The end result is

$$\frac{1}{2}\frac{d^2 I^{jk}}{dt^2} = 2\mathcal{T}^{jk} + \Omega^{jk} + P\delta^{jk} \,, \tag{1.88}$$

the statement of the *tensor virial theorem*. Taking the trace of Eq. (1.88) returns

$$\frac{1}{2}\frac{d^2 I}{dt^2} = 2\mathcal{T} + \Omega + 3P \,, \tag{1.89}$$

the *scalar virial theorem*; here $I(t) := \int \rho r^2 \, d^3x$ is the trace of the quadrupole moment tensor.

Many applications of the virial theorems involve stationary systems, for which $d^2 I^{jk}/dt^2 = 0$. For such systems the virial theorems reduce to

$$2\mathcal{T}^{jk} + \Omega^{jk} + P\delta^{jk} = 0 \,, \tag{1.90a}$$

$$2\mathcal{T} + \Omega + 3P = 0 \,. \tag{1.90b}$$

Other applications involve periodic systems, for which we may integrate Eqs. (1.88) and (1.89) over a complete period of the system. In these circumstances the terms involving the quadrupole moment tensor disappear also, and Eqs. (1.90) continue to hold in a coarse-grained form; the equations now involve *averages* of $\mathcal{T}^{jk}$, Ω^{jk}, and P over a period of the system.

The virial theorems are powerful tools, and they can be exploited to great benefits in the study of stellar structure. In the context of this book we find them most useful in our study of post-Newtonian equations of motion (in Chapter 9) and gravitational waves (in Chapter 11).

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Box 1.4 Symmetrized and antisymmetrized indices

We define symmetrized and anti-symmetrized indices according to

$$A^{(j}B^{k)} := \frac{1}{2}\big(A^j B^k + A^k B^j\big)\,,$$
$$A^{[j}B^{k]} := \frac{1}{2}\big(A^j B^k - A^k B^j\big)\,.$$

These definitions apply equally well to tensors, for example $C^{(jk)} = \frac{1}{2}(C^{jk} + C^{kj})$ and $C^{[jk]} = \frac{1}{2}(C^{jk} - C^{kj})$. Tensors of higher ranks can also be symmetrized and antisymmetrized in an obvious way, so that a symmetrized rank-q tensor is defined by

$$C^{(k_1 k_2 \cdots k_q)} := \frac{1}{q!}\big(C^{k_1 k_2 \cdots k_q} + \cdots\big)\,,$$

where the remaining terms consist of all possible permutations of the q indices. An antisymmetrized rank-q tensor is defined by

$$C^{[k_1 k_2 \cdots k_q]} := \frac{1}{q!}\big(C^{k_1 k_2 \cdots k_q} \pm \cdots\big)\,,$$

where the sign of each term is positive when the index order is an even permutation of the original order, and negative when it is an odd permutation.

1.4.4 Mass–momentum tensor

An alternative formulation of the equations of hydrodynamics reveals a nice parallel with the relativistic equations to be introduced in Chapters 4 and 5. This formulation is based on the Eulerian (as opposed to Lagrangian) version of these equations, as given by Eqs. (1.24) and (1.26):

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0\,, \tag{1.91a}$$

$$\rho\left(\frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{v}\cdot\nabla\boldsymbol{v}\right) = \rho\nabla U - \nabla p\,. \tag{1.91b}$$

It involves a repackaging of the main fluid variables, such as the mass density ρ and the momentum density $\rho\boldsymbol{v}$, into a *mass–momentum tensor* with components T^{tt}, T^{tj}, T^{jt}, and T^{jk}.

To elaborate the new formulation it is helpful to work in terms of vector components (such as x^j) instead of the vectors themselves. In this language, for example, we would write $\nabla\cdot\boldsymbol{v}$ as $\sum_j(\partial v^j/\partial x^j)$. To save a lot of unnecessary writing we adopt an important convention that was first introduced by Einstein in his papers on general relativity (he jokingly considered it to be a great mathematical discovery). According to the *Einstein summation convention*, we omit the summation symbol whenever two indices are repeated, and automatically sum the indices (such as j with j) over the full range of their values – in this case x, y, and z, or more generally, over all three spatial dimensions. Thus we write $\nabla \cdot \boldsymbol{v}$ simply as $\partial v^j/\partial x^j$.

In the component language the continuity equation (1.91a) takes the form of

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x^j}(\rho v^j) = 0\,, \tag{1.92}$$

and we can also show, by making use of the continuity equation, that the left-hand side of Eq. (1.91b) can be written as

$$\frac{\partial}{\partial t}(\rho v^j) + \frac{\partial}{\partial x^k}(\rho v^j v^k)\,.$$

To recast the right-hand side of Eq. (1.91b) we invoke Poisson's equation (1.15) and express ρ in terms of $\nabla^2 U$; this yields

$$\rho \frac{\partial U}{\partial x^j} = -\frac{1}{4\pi G}\nabla^2 U \frac{\partial U}{\partial x^j}\,, \tag{1.93}$$

which can be re-expressed as

$$\rho \frac{\partial U}{\partial x^j} = -\frac{1}{4\pi G}\frac{\partial}{\partial x^k}\left(\frac{\partial U}{\partial x^j}\frac{\partial U}{\partial x^k} - \frac{1}{2}\delta_{jk}\frac{\partial U}{\partial x^p}\frac{\partial U}{\partial x^p}\right)\,. \tag{1.94}$$

To verify the equality of these two expressions makes a worthy exercise in index manipulation. Recall that according to the summation convention, $(\partial U/\partial x^p)(\partial U/\partial x^p) = \nabla U \cdot \nabla U = |\nabla U|^2$. Continuing our work on the right-hand side of Eq. (1.91b), we note that $\partial p/\partial x^j$ can be written as $\partial(p\delta^{jk})/\partial x^k$.

Collecting these results, we find that we may write the continuity and Euler equations in the compact form

$$\frac{\partial}{\partial t}T^{tt} + \frac{\partial}{\partial x^k}T^{tk} = 0\,, \tag{1.95a}$$

$$\frac{\partial}{\partial t}T^{jt} + \frac{\partial}{\partial x^k}T^{jk} = 0\,, \tag{1.95b}$$

where the components of the mass–momentum tensor are given by

$$T^{tt} := \rho\,, \tag{1.96a}$$

$$T^{tj} := \rho v^j\,, \tag{1.96b}$$

$$T^{jt} := \rho v^j\,, \tag{1.96c}$$

$$T^{jk} := \rho v^j v^k + p\,\delta^{jk} + \frac{1}{4\pi G}\left(\frac{\partial U}{\partial x^j}\frac{\partial U}{\partial x^k} - \frac{1}{2}\delta_{jk}\frac{\partial U}{\partial x^p}\frac{\partial U}{\partial x^p}\right)\,. \tag{1.96d}$$

Notice that the tensor is symmetric under an exchange of any pair of indices.

It is worthwhile to introduce yet more notation to simplify the appearance of Eqs. (1.95). We shall henceforth denote the partial derivatives of a function f by

$$\partial_t f := \frac{\partial f}{\partial t}\,, \qquad \partial_j f := \frac{\partial f}{\partial x^j}\,. \tag{1.97}$$

In addition, we shall use Greek indices (such as α and β) to denote all four space and time variables; Greek indices run over the values t, x, y, and z, while Latin indices continue to run over the three spatial values. In this notation, the partial derivatives of Eq. (1.97) are collectively denoted $\partial_\alpha f$. To reflect this new usage, we shall extend the Einstein summation convention so that it applies also to repeated Greek indices; here summation will run over the four spacetime values, while summation over repeated Latin indices will continue to run over the three spatial values.

With these new rules, the continuity and Euler equations (1.95) can be written in the wonderfully compact form

$$\partial_\beta T^{\alpha\beta} = 0 . \tag{1.98}$$

This is a set of four distinct equations, and we stress that summation of β extends over the four values t, x, y, and z. Some comments are in order. The first is an admission of humility: while we have managed to combine space and time indices into a seemingly unified expression, we hasten to point out that there is absolutely nothing relativistic about this. What we have here is a mere repackaging of the Newtonian equations of hydrodynamics, in a compact form that happens to anticipate the relativistic versions of Chapters 4 and 5. The second comment is a subtle one of interpretation. The original continuity and Euler equations, Eqs. (1.5) and (1.6), were derived from basic principles of Newtonian mechanics, and they would continue to hold even if Eq. (1.15), the equation governing the behavior of the gravitational potential, turned out to be invalid. By contrast, Poisson's equation was involved in the derivation of Eq. (1.98), and its validity therefore rests on the validity of the Newtonian field equation. This reveals that in the context of Newtonian mechanics, Eq. (1.98) is not quite as fundamental or general as the continuity and Euler equations. We shall see in Chapter 5 that the point of view is very different in a relativistic context; there it is Eq. (1.98) – or a suitable generalization thereof – that forms the fundamental starting point of a derivation of the relativistic continuity and Euler equations; in particular, we shall see that Eq. (1.98) is quite independent from, and indeed more fundamental than, the Einstein field equations.

Writing the equations of hydrodynamics in the form of Eqs. (1.95) or (1.98) suggests a very efficient way of deriving the conservation statements regarding mass, momentum, angular momentum, and center-of-mass motion for an isolated system. We first introduce new definitions for these quantities:

$$M := \int T^{tt}\, d^3x , \tag{1.99a}$$

$$P^j := \int T^{jt}\, d^3x , \tag{1.99b}$$

$$R^j := \frac{1}{M} \int T^{tt} x^j\, d^3x , \tag{1.99c}$$

$$J^j := \epsilon^{jpq} \int x^p T^{qt}\, d^3x , \tag{1.99d}$$

where ϵ^{jpq} is the completely anti-symmetric Levi–Civita symbol, whose value is 1 if jpq is an even permutation of 123 (or xyz), -1 if it is an odd permutation, and zero if any two indices take on the same value. The Levi–Civita symbol is a convenient tool for constructing cross products when using the index language; you may easily check that the components

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of $\boldsymbol{a} \times \boldsymbol{b}$ are $\epsilon^{jpq} a^p b^q$, if you keep in mind the implied summation of p and q. The new definitions of Eqs. (1.99) are fully equivalent to the old definitions of Eqs. (1.53), (1.54), (1.57), and (1.78).

It is then a simple matter to take the time derivative of each one of these quantities, to exploit the fact that according to Eq. (1.95), a time derivative of T^{tt} or T^{jt} is equal to a spatial divergence, and to use Gauss's theorem to re-express the volume integral as a surface integral; because T^{tj} or T^{jk} vanishes on the surface, this integral vanishes, and the quantity is conserved. (This statement is true only when the domain of integration V extends over all space, and the boundary S is situated at infinity. Unlike our previous usage in Sec. 1.4.3, here it is not sufficient for S to merely enclose the fluid system. The reason is that while ρ and p can be trusted to vanish outside the matter, the Newtonian potential U cannot; instead it falls off as GM/r far away from the fluid system. To ensure that the surface integral associated with T^{jk} vanishes, it is necessary to place S at infinity, where $\partial_j U$ properly vanishes.) The end result of this efficient computation is the statement that M, $\boldsymbol{P}$, and $\boldsymbol{J}$ are constant, and that $d\boldsymbol{R}/dt = \boldsymbol{P}/M$. You will be asked to go through the steps of these computations in Exercise 1.5. It is interesting to observe that the proof of the constancy of $\boldsymbol{J}$ relies critically on the fact that T^{jk} is symmetric in its indices; the detailed expression for T^{jk} is never required.

Another result that follows by repeated use of Eq. (1.98) and Gauss's theorem is

$$\frac{1}{2}\frac{d^2}{dt^2}\int T^{tt} x^j x^k \, d^3x = \int T^{jk} \, d^3x \,. \tag{1.100}$$

Because $\int T^{tt} x^j x^k \, d^3x$ is the quadrupole moment tensor of the mass distribution, this is an alternative statement of the tensor virial theorem, which can be shown to be equivalent to the statement given previously in Eq. (1.88). You will be asked to go through the steps of this computation in Exercise 1.6.

It is important to appreciate that the formulation of the equations of motion using a mass–momentum tensor *does not* yield an expression for the conserved energy; for that we must follow our earlier derivation and deal directly with Euler's equation. In a relativistic formulation of the equations, however, the statement of energy conservation would also follow, and would in fact supersede the statement of mass conservation. The reason, of course, is that in a relativistic context, mass is but a form of energy, to be naturally included in a breakdown such as Eq. (1.67). In a relativistic context, therefore, mass conservation arises as a special case of energy conservation. In Newtonian mechanics conservation of mass is necessarily separated from conservation of energy.

1.5 Spherical and nearly spherical bodies

We next consider the problem of calculating the gravitational potential U for various kinds of bodies. The simplest case has already been dealt with: as we have seen, the potential of a single point mass M at the origin of the coordinate system is given by $U = GM/r$. We wish to go well beyond this simplest case, and to construct the Newtonian potential for a

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more realistic, finite-sized body with an arbitrary mass density ρ. A body can often be taken to be spherically symmetric as a first approximation, and we shall begin our presentation with a complete description of this idealized case. This is not sufficient, however, because deviations from spherical symmetry can often be very important. We shall therefore devote the remainder of this section to a description of non-spherical bodies. While our treatment will be exact, the formalism that we develop – based on a multipole expansion of the mass distribution and the gravitational potential – is most powerful when applied to bodies that deviate only modestly from spherical symmetry.

For our purposes in this section it is best to express the Laplacian operator in spherical polar coordinates (r, θ, ϕ), and to write Poisson's equation (1.15) as

$$\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial}{\partial r} + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{r^2\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right)U = -4\pi G\rho\,, \tag{1.101}$$

in which ρ and U are functions of t, r, θ, and ϕ. We recall that the relation to Cartesian coordinates is given by $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, and $z = r\cos\theta$.

1.5.1 Spherical bodies

The mass density ρ and gravitational potential U of a spherical body depend on t and r only, and in this case Eq. (1.101) reduces to

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) = -4\pi G\rho(t, r)\,. \tag{1.102}$$

Integrating once, we obtain

$$\frac{\partial U}{\partial r} = -\frac{4\pi G}{r^2}\int_0^r \rho(t, r')r'^2\,dr'\,, \tag{1.103}$$

where the constant of integration (actually a function of t) was chosen so that the gravitational force at $r = 0$ vanishes, as it must by symmetry. To ensure this we make the physically reasonable requirement that the density ρ must be finite at $r = 0$; then the integral behaves as $\frac{1}{3}\rho(t, 0)r^3$ near $r = 0$, and $\partial U/\partial r$ properly vanishes at $r = 0$.

The integral in Eq. (1.103) defines the mass contained inside a sphere of radius r, which we write as

$$m(t, r) := \int_0^r 4\pi\rho(t, r')r'^2\,dr'\,. \tag{1.104}$$

The mass density drops to zero at the surface of the body, and the body's total mass is

$$M := m(t, r = R) = \int_0^R 4\pi\rho(t, r')r'^2\,dr'\,, \tag{1.105}$$

with R denoting the body's radius. It was already established in Sec. 1.4.3 that M does not depend on time, in spite of the fact that $m(t, r)$ itself may depend on time. For example, a pulsating star would display a time-dependent density ρ, a time-dependent internal mass

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function m, and a time-dependent stellar radius R, but its total mass M would still be constant.

Returning to Eq. (1.103), we find that the gravitational force acting on a fluid element of unit mass situated at radius r is given by

$$\frac{\partial U}{\partial r} = -\frac{Gm(t,r)}{r^2} . \tag{1.106}$$

We observe that in spherical symmetry, the force is completely determined by the mass contained inside the sphere of radius r, and that any spherical distribution of matter outside this sphere contributes nothing to the force. While this result is an almost trivial consequence of integrating Poisson's equation, it was far from obvious to Newton while he was struggling to prove it with the traditional geometrical methods that he adopted throughout the *Principia*. He describes this struggle in a commentary that follows Proposition 8 in Book 2:

> "After I had found that the gravity toward a whole planet arises from and is compounded of the gravities toward the parts and that toward each of the individual parts it is inversely proportional to the squares of the distances from the parts, I was still not certain whether that proportion of the inverse square obtained exactly in a total force compounded of a number of forces, or only nearly so. For it could happen that a proportion which holds exactly enough at very great distances might be markedly in error near the surface of the planet, because there the distances of the particles may be unequal and their situations dissimilar. But at length, by means of Book 1, Propositions 75 and 76 and their corollaries, I discerned the truth of the proposition dealt with here."

In Propositions 75 and 76, Newton proves that spherical bodies attract each other with a force inversely proportional to the square of the distance between their centers; previously, in Proposition 71, he proved that the force on a particle outside a spherical distribution of matter is given by Eq. (1.106).

Outside the spherical body, $m(t,r) = M$ and Eq. (1.106) becomes

$$\frac{dU}{dr} = -\frac{GM}{r^2} . \tag{1.107}$$

The gravitational force is now time-independent and completely determined by the total mass of the body. These statements apply to the potential itself: the gravitational potential outside a spherical body is constant in time regardless of the time-dependence of the matter distribution, and determined by the body's total mass. As we shall see in Chapter 5, a similar statement can be made in general relativity, where it is known as *Birkhoff's theorem*.

The potential itself is determined by integrating Eqs. (1.106) and (1.107). We impose continuity of U at $r = R$ and the boundary condition that U should vanish at $r = \infty$. We obtain

$$U(t,r) = \frac{GM}{R} + G\int_r^R \frac{m(t,r')}{r'^2}\,dr' \tag{1.108}$$

inside the matter (for $r < R$), and

$$U(r) = \frac{GM}{r} \tag{1.109}$$

outside the matter (for $r > R$). Integration by parts reveals that an equivalent expression for the internal potential is

$$U(t, r) = \frac{Gm(t, r)}{r} + 4\pi G \int_r^R \rho(t, r')r'\, dr' \,. \tag{1.110}$$

This expression can be seen to apply outside the matter as well, where $\rho = 0$ and $m = M$. The potential at the center of the body is given by $U(t, 0) = 4\pi G \int_0^R \rho(t, r')r'\, dr'$.

1.5.2 Non-spherical bodies

Multipole expansions

Spherical symmetry is a convenient simplification, but real bodies are seldom spherically symmetric. Rotation, interactions with other bodies, and stresses caused by solid materials (such as crusts in neutron stars) can lead to deviations from spherical symmetry. For bodies such as planets or stars, these deviations are usually small. This makes the method of *multipole expansions* a useful and powerful tool in modelling the gravitational field of such objects. If the deviations from spherical symmetry are small, the contributions to the gravitational potential of higher multipole moments of the mass distribution are progressively smaller, so that often a small number of moments is sufficient to give an accurate description of the gravitational field for most problems of interest.

Conversely, the determination of the multipole moments of a body by a precise measurement of its external field can supply important diagnostic information about its interior. Indeed, *geodesy* is the science of determining the Earth's gravitational field to high precision as a means of understanding the Earth's internal structure and dynamics. Modern geodetic measurements using precise tracking of Earth-orbiting satellites are determining the Earth's multipole moments up to $\ell = 360$, corresponding to variations in the Earth's surface gravity on a scale of 100 km. From these data it is possible to study such phenomena as the ongoing rebound of the continents following the disappearance of ice-age glaciers, seasonal variations of the amount of water in the Amazon basin, and the effects of volcanism and continental drift.

Some bodies, such as disk-shaped spiral galaxies, are not close to being spherical, and in such cases the method of multipole expansions must be relinquished in favor of other methods, mainly numerical, to solve for the gravitational potential. In the laboratory, for example, experiments to measure the gravitational constant G or to test the weak equivalence principle must carefully account for stray gravitational forces caused by nearby laboratory apparatus. This involves the determination of the gravitational field of strangely shaped objects, often including wires and knobs. The masses and shapes of all these objects must be measured precisely, and the gravitational field calculated

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numerically, often using techniques borrowed from engineering, such as "finite element" methods.

Spherical-harmonic decomposition

We shall focus our attention on a nearly spherical body, and provide a description of its gravitational field in terms of a multipole expansion. The key analytical tool to achieve this is a systematic expansion of our main variables, ρ and U, in terms of *spherical-harmonic functions* $Y_{\ell m}(\theta, \phi)$, in which ℓ is an integer that ranges from 0 to ∞, while m is a second integer that ranges from $-\ell$ to ℓ for each value of ℓ; the polar angles θ and ϕ were introduced previously in Eq. (1.101).

We recall that the spherical harmonics are solutions to the eigenvalue equation

$$\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\sin\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin^2\theta}\frac{\partial^2}{\partial\phi^2}\right) Y_{\ell m} = -\ell(\ell+1)Y_{\ell m}\,, \tag{1.111}$$

in which the left-hand side is recognized as the angular piece of the Laplacian operator in Eq. (1.101). For $m = 0$ they are given explicitly by

$$Y_{\ell 0}(\theta) = \sqrt{\frac{2\ell+1}{4\pi}}\, P_\ell(\cos\theta)\,, \tag{1.112}$$

where

$$P_\ell(\mu) := \frac{1}{2^\ell \ell!}\frac{d^\ell}{d\mu^\ell}(\mu^2-1)^\ell \tag{1.113}$$

are the well-known *Legendre polynomials*; in this case the spherical harmonics are independent of ϕ. For $m > 0$ they are given explicitly by

$$Y_{\ell m}(\theta,\phi) = \sqrt{\frac{2\ell+1}{4\pi}\frac{(\ell-m)!}{(\ell+m)!}}\, P^m_\ell(\cos\theta)e^{im\phi}\,, \tag{1.114}$$

where

$$P^m_\ell(\mu) := (-1)^m(1-\mu^2)^{m/2}\frac{d^m}{d\mu^m}P_\ell(\mu) \tag{1.115}$$

are the *associated Legendre functions*. For $m < 0$ we use the formula

$$Y_{\ell,-m}(\theta,\phi) = (-1)^m Y^*_{\ell m}(\theta,\phi)\,. \tag{1.116}$$

The spherical harmonics form a set of orthogonal functions, and they are normalized so that

$$\int Y_{\ell m}(\theta,\phi)Y^*_{\ell' m'}(\theta,\phi)\, d\Omega = \delta_{\ell\ell'}\delta_{mm'}\,, \tag{1.117}$$

where $d\Omega := \sin\theta\, d\theta d\phi$ is an element of solid angle in the direction specified by θ and ϕ; the integral extends over the entire two-sphere (any surface $r =$ constant), from $\phi = 0$ to $\phi = 2\pi$, and from $\theta = 0$ to $\theta = \pi$. The spherical harmonics also form a *complete set*

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of orthonormal functions, meaning that any square-integrable function on the two-sphere – any well-behaved function $f(\theta, \phi)$ – can be expanded as

$$f(\theta, \phi) = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} f_{\ell m} Y_{\ell m}(\theta, \phi) , \tag{1.118}$$

with coefficients given by

$$f_{\ell m} = \int f(\theta, \phi) Y^*_{\ell m}(\theta, \phi)\, d\Omega . \tag{1.119}$$

The spherical-harmonic functions of order $\ell = \{0, 1, 2, 3\}$ are listed in Box 1.5.

Box 1.5 **Spherical harmonics**

The spherical-harmonic functions of lowest order are given explicitly by

$$Y_{00} = \frac{1}{\sqrt{4\pi}} ,$$
$$Y_{10} = \sqrt{\frac{3}{4\pi}} \cos\theta ,$$
$$Y_{11} = -\sqrt{\frac{3}{8\pi}} \sin\theta\, e^{i\phi} ,$$
$$Y_{20} = \sqrt{\frac{5}{16\pi}} (3\cos^2\theta - 1) ,$$
$$Y_{21} = -\sqrt{\frac{15}{8\pi}} \sin\theta \cos\theta\, e^{i\phi} ,$$
$$Y_{22} = \sqrt{\frac{15}{32\pi}} \sin^2\theta\, e^{2i\phi} ,$$
$$Y_{30} = \sqrt{\frac{7}{16\pi}} (5\cos^3\theta - 3\cos\theta) ,$$
$$Y_{31} = -\sqrt{\frac{21}{64\pi}} \sin\theta (5\cos^2\theta - 1)\, e^{i\phi} ,$$
$$Y_{32} = \sqrt{\frac{105}{32\pi}} \sin^2\theta \cos\theta\, e^{2i\phi} ,$$
$$Y_{33} = -\sqrt{\frac{35}{64\pi}} \sin^3\theta\, e^{3i\phi} .$$

As explained in the main text (in Sec. 1.5.3), the spherical harmonics can be expressed as an expansion in symmetric-tracefree (STF) tensors, according to

$$Y_{\ell m}(\theta, \phi) = \mathscr{Y}^{*\langle L\rangle}_{\ell m} n_{\langle L\rangle} ,$$

in which $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ is a constant STF tensor, while $n_{\langle L\rangle}$ is a STF combination of unit radial vectors, with $\boldsymbol{n} := [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta]$. Specific examples are

$$\mathscr{Y}^{\langle z\rangle}_{10} = \sqrt{\frac{3}{4\pi}},$$

$$\mathscr{Y}^{\langle x\rangle}_{11} = -\sqrt{\frac{3}{8\pi}}, \qquad \mathscr{Y}^{\langle y\rangle}_{11} = i\sqrt{\frac{3}{8\pi}},$$

$$\mathscr{Y}^{\langle xx\rangle}_{20} = -\sqrt{\frac{5}{16\pi}}, \qquad \mathscr{Y}^{\langle yy\rangle}_{20} = -\sqrt{\frac{5}{16\pi}}, \qquad \mathscr{Y}^{\langle zz\rangle}_{20} = 2\sqrt{\frac{5}{16\pi}},$$

$$\mathscr{Y}^{\langle xz\rangle}_{21} = -\frac{1}{2}\sqrt{\frac{15}{8\pi}}, \qquad \mathscr{Y}^{\langle yz\rangle}_{21} = \frac{i}{2}\sqrt{\frac{15}{8\pi}},$$

$$\mathscr{Y}^{\langle xx\rangle}_{22} = \sqrt{\frac{15}{32\pi}}, \qquad \mathscr{Y}^{\langle xy\rangle}_{22} = -i\sqrt{\frac{15}{32\pi}}, \qquad \mathscr{Y}^{\langle yy\rangle}_{22} = -\sqrt{\frac{15}{32\pi}},$$

$$\mathscr{Y}^{\langle xxz\rangle}_{30} = -\sqrt{\frac{7}{16\pi}}, \qquad \mathscr{Y}^{\langle yyz\rangle}_{30} = -\sqrt{\frac{7}{16\pi}}, \qquad \mathscr{Y}^{\langle zzz\rangle}_{30} = 2\sqrt{\frac{7}{16\pi}},$$

$$\mathscr{Y}^{\langle xxx\rangle}_{31} = \sqrt{\frac{21}{64\pi}}, \qquad \mathscr{Y}^{\langle xxy\rangle}_{31} = -\frac{i}{3}\sqrt{\frac{21}{64\pi}}, \qquad \mathscr{Y}^{\langle xyy\rangle}_{31} = \frac{1}{3}\sqrt{\frac{21}{64\pi}},$$

$$\mathscr{Y}^{\langle xzz\rangle}_{31} = -\frac{4}{3}\sqrt{\frac{21}{64\pi}}, \qquad \mathscr{Y}^{\langle yzz\rangle}_{31} = \frac{4i}{3}\sqrt{\frac{21}{64\pi}}, \qquad \mathscr{Y}^{\langle yyy\rangle}_{31} = -i\sqrt{\frac{21}{64\pi}},$$

$$\mathscr{Y}^{\langle xxz\rangle}_{32} = \frac{1}{3}\sqrt{\frac{105}{32\pi}}, \qquad \mathscr{Y}^{\langle xyz\rangle}_{32} = -\frac{i}{3}\sqrt{\frac{105}{32\pi}}, \qquad \mathscr{Y}^{\langle yyz\rangle}_{32} = -\frac{1}{3}\sqrt{\frac{105}{32\pi}},$$

$$\mathscr{Y}^{\langle xxx\rangle}_{33} = -\sqrt{\frac{35}{64\pi}}, \quad \mathscr{Y}^{\langle xxy\rangle}_{33} = i\sqrt{\frac{35}{64\pi}}, \quad \mathscr{Y}^{\langle xyy\rangle}_{33} = \sqrt{\frac{35}{64\pi}}, \quad \mathscr{Y}^{\langle yyy\rangle}_{33} = -i\sqrt{\frac{35}{64\pi}}.$$

Only the independent, non-vanishing components are listed, and other components can be obtained by exploiting the index symmetries of $\mathscr{Y}^{\langle L\rangle}_{\ell m}$.

Reduced Poisson equation

To integrate Eq. (1.101) we decompose the mass density ρ and Newtonian potential U in spherical harmonics,

$$\rho(t,r,\theta,\phi) = \sum_{\ell m} \rho_{\ell m}(t,r) Y_{\ell m}(\theta,\phi), \tag{1.120a}$$

$$U(t,r,\theta,\phi) = \sum_{\ell m} U_{\ell m}(t,r) Y_{\ell m}(\theta,\phi), \tag{1.120b}$$

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in which the coefficients

$$\rho_{\ell m}(t,r) = \int \rho(t,r,\theta,\phi) Y^*_{\ell m}(\theta,\phi)\, d\Omega\,, \tag{1.121a}$$

$$U_{\ell m}(t,r) = \int U(t,r,\theta,\phi) Y^*_{\ell m}(\theta,\phi)\, d\Omega \tag{1.121b}$$

are now allowed to depend on t and r; in Eq. (1.120) the summation sign is a shorthand notation for the double sum that appears in Eq. (1.118). Making the substitution in Eq. (1.101) produces the decoupled set of ordinary differential equations

$$\mathscr{L} U_{\ell m} = -4\pi G r^2 \rho_{\ell m}\,, \tag{1.122}$$

in which

$$\mathscr{L} := \frac{\partial}{\partial r} r^2 \frac{\partial}{\partial r} - \ell(\ell+1) \tag{1.123}$$

is an effective Laplacian operator (multiplied by r^2) that involves the radial coordinate only. Equation (1.122) is a dimensionally-reduced version of Poisson's equation, and we have simplified the problem of finding a solution to a partial differential equation in three variables to that of finding solutions to an infinite number of ordinary differential equations.

It is helpful here, as it was in Sec. 1.3, to integrate Eq. (1.122) with the help of a Green's function $g_\ell(r,r')$, which is required to be a solution to

$$\mathscr{L} g_\ell(r,r') = -4\pi\delta(r-r')\,. \tag{1.124}$$

It is easy to verify that once the Green's function is available, $U_{\ell m}$ can be obtained for any $\rho_{\ell m}$ by evaluating the integral

$$U_{\ell m}(t,r) = G \int g_\ell(r,r') \rho_{\ell m}(t,r') r'^2\, dr'\,. \tag{1.125}$$

The Green's function is not difficult to construct. First we observe that if $U_<(r)$ and $U_>(r)$ are independent solutions to the homogeneous equation $\mathscr{L}U = 0$, then

$$g(r,r') = U_<(r)\Theta(r'-r) + U_>(r)\Theta(r-r') \tag{1.126}$$

is a solution to Eq. (1.124) provided that $U_>(r') - U_<(r') = 0$ and $U'_>(r') - U'_<(r') = -4\pi/r'^2$. We omit the label ℓ to simplify the notation, and a prime on U indicates differentiation with respect to r; $\Theta(r-r')$ is the Heaviside step function, equal to one when $r - r' > 0$ and zero otherwise, and such that $\Theta'(r-r') = \delta(r-r')$. For $U_<(r)$ we choose a solution that is finite at $r = 0$, and this requirement forces $U_< \propto r^\ell$. For $U_>(r)$ we choose a solution that is finite at $r = \infty$, and this requirement forces $U_> \propto r^{-(\ell+1)}$. The junction conditions at $r = r'$ determine the constants of proportionality (which actually depend on r'), and we finally arrive at

$$g_\ell(r,r') = \frac{4\pi}{2\ell+1}\left[\frac{r^\ell}{r'^{\ell+1}}\Theta(r'-r) + \frac{r'^\ell}{r^{\ell+1}}\Theta(r-r')\right] = \frac{4\pi}{2\ell+1}\frac{r_<^\ell}{r_>^{\ell+1}}\,, \tag{1.127}$$

where $r_< := \min(r,r')$ and $r_> := \max(r,r')$.

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Substituting Eq. (1.127) in Eq. (1.125) returns

$$U_{\ell m}(t,r) = \frac{4\pi G}{2\ell+1}\biggl[r^\ell \int_r^\infty \frac{\rho_{\ell m}(t,r')}{r'^{\ell+1}} r'^2\, dr' + \frac{1}{r^{\ell+1}} \int_0^r r'^\ell \rho_{\ell m}(t,r') r'^2\, dr' \biggr], \tag{1.128}$$

and this expression is ready to be inserted within Eq. (1.120). We express our final result as

$$U(t,r,\theta,\phi) = G \sum_{\ell m} \frac{4\pi}{2\ell+1}\biggl[q_{\ell m}(t,r) \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}} + p_{\ell m}(t,r) r^\ell Y_{\ell m}(\theta,\phi) \biggr], \tag{1.129}$$

where

$$q_{\ell m}(t,r) := \int_0^r r'^\ell \rho_{\ell m}(t,r') r'^2\, dr' \tag{1.130}$$

and

$$p_{\ell m}(t,r) := \int_r^R \frac{\rho_{\ell m}(t,r')}{r'^{\ell+1}} r'^2\, dr'\,. \tag{1.131}$$

These relations apply directly to the body's interior, in which $\rho_{\ell m} \neq 0$, and the integral defining $p_{\ell m}$ was truncated to a sphere of arbitrary radius R that surrounds the matter distribution. They apply also to the body's exterior, where $\rho_{\ell m} = 0$, but here they simplify to

$$U_{\rm ext}(t,r,\theta,\phi) = G \sum_{\ell m} \frac{4\pi}{2\ell+1} q_{\ell m}(t,R) \frac{Y_{\ell m}(\theta,\phi)}{r^{\ell+1}}\,; \tag{1.132}$$

the term involving $p_{\ell m}$ vanishes, and $q_{\ell m}$ was evaluated at $r = R$, because the integrals of Eqs. (1.130) and (1.131) are now evaluated outside the matter distribution.

Integral solution

The results of Eqs. (1.129) and(1.132) can be reproduced by proceeding directly from the integral solution to Poisson's equation,

$$U(t,\boldsymbol{x}) = G \int \frac{\rho(t,\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x'\,. \tag{1.133}$$

The strategy is to express $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$, the three-dimensional Green's function of Eq. (1.18), as the spherical-harmonic decomposition

$$\frac{1}{|\boldsymbol{x}-\boldsymbol{x}'|} = \sum_{\ell m} \frac{4\pi}{2\ell+1} \frac{r_<^\ell}{r_>^{\ell+1}} Y^*_{\ell m}(\theta',\phi') Y_{\ell m}(\theta,\phi)\,, \tag{1.134}$$

in which (r,θ,ϕ) are the spherical polar coordinates of the field point $\boldsymbol{x}$, while (r',θ',ϕ') are the coordinates of the source point $\boldsymbol{x}'$; we recall that $r_< := \min(r,r')$ and $r_> := \max(r,r')$. We recognize the radial Green's function $g_\ell(r,r')$ in the first factor within the sum in Eq. (1.134), and the product of spherical harmonics accounts for the angular dependence.

When we insert Eq. (1.134) within $U(t,\boldsymbol{x})$ and evaluate the integral outside the matter distribution, we find that $r_< = r'$ and $r_> = r$ because the variable of integration r' is always

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smaller than r; we end up once more with Eq. (1.132), with $q_{\ell m}(t, R)$ given by Eq. (1.130) and $\rho_{\ell m}$ given by Eq. (1.121). When instead we evaluate the integral inside the matter, the radial integration must be broken up into two pieces, the first ranging from $r' = 0$ to $r' = r$, for which $r_< = r'$ and $r_> = r$, and the second ranging from $r' = r$ to $r' = R$, for which $r_< = r$ and $r_> = r'$; we end up with Eq. (1.129), with $q_{\ell m}(t, r)$ and $p_{\ell m}(t, r)$ given by Eqs. (1.130) and (1.131), respectively.

To establish Eq. (1.134) we make use of two fundamental properties of the Legendre polynomials. The first is that the polynomials come with a *generating function*

$$\frac{1}{\sqrt{1 - 2\eta\mu + \eta^2}} = \sum_{\ell=0}^{\infty} \eta^\ell P_\ell(\mu)\,, \tag{1.135}$$

in which η is an arbitrary number smaller than unity, and μ is the argument of the Legendre functions. In this representation, $P_\ell(\mu)$ is recognized as the set of coefficients in a Taylor expansion of the generating function in powers of η. In our particular application η is identified with $r_</r_>$, μ is identified with $\cos\gamma := \boldsymbol{x} \cdot \boldsymbol{x}'/(rr')$, and we identify the generating function with $r_>|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$. Equation (1.135) yields

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \sum_{\ell=0}^{\infty} \frac{r_<^\ell}{r_>^{\ell+1}} P_\ell(\cos\gamma)\,. \tag{1.136}$$

The second property is the *addition theorem*

$$P_\ell(\cos\gamma) = \frac{4\pi}{2\ell+1} \sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta', \phi') Y_{\ell m}(\theta, \phi)\,, \tag{1.137}$$

in which γ is related to the angles (θ, ϕ) and (θ', ϕ') by

$$\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos(\phi - \phi')\,. \tag{1.138}$$

Inserting Eq. (1.137) in Eq. (1.136) returns Eq. (1.134).

Multipole moments

The quantities $I_{\ell m}(t) := q_{\ell m}(t, R)$, as defined by Eqs. (1.121) and (1.130), are the *multipole moments* of the mass distribution. Their full expression is

$$I_{\ell m}(t) := \int \rho(t, \boldsymbol{x}) r^\ell Y^*_{\ell m}(\theta, \phi)\, d^3x\,, \tag{1.139}$$

where the domain of integration extends over the volume occupied by the matter. (Because the domain is independent of the field point $\boldsymbol{x}$, it is no longer necessary to adorn the variables of integration with primes.) In terms of these the external potential is given by

$$U_{\text{ext}}(t, \boldsymbol{x}) = G \sum_{\ell m} \frac{4\pi}{2\ell+1} I_{\ell m}(t) \frac{Y_{\ell m}(\theta, \phi)}{r^{\ell+1}}\,; \tag{1.140}$$

this expression is copied directly from Eq. (1.132) – this is such an important result, we feel compelled to display it twice.

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The moment corresponding to $\ell = m = 0$ is known as the *monopole moment*, and it is intimately related to the body's total mass:

$$I_{00} = \int \rho Y_{00}\, d^3x = \frac{M}{\sqrt{4\pi}}\,. \tag{1.141}$$

The moments corresponding to $\ell = 1$ (and $m = \{-1, 0, 1\}$) are known as the *dipole moments*. These all vanish when we place the origin of the coordinate system at the body's center-of-mass, so that $\int \rho \boldsymbol{x}\, d^3x = \boldsymbol{0}$. This conclusion follows from a simple computation:

$$I_{10} = \sqrt{\frac{3}{4\pi}} \int \rho\, r \cos\theta\, d^3x = \sqrt{\frac{3}{4\pi}} \int \rho z\, d^3x = 0\,, \tag{1.142a}$$

$$I_{1\pm1} = \mp\sqrt{\frac{3}{8\pi}} \int \rho\, r \sin\theta e^{\pm i\phi}\, d^3x = \mp\sqrt{\frac{3}{8\pi}} \int \rho(x \pm iy)\, d^3x = 0\,. \tag{1.142b}$$

Moments of higher degree are called quadrupole ($\ell = 2$), octopole ($\ell = 3$), hexadecapole ($\ell = 4$), and so on. When the body is spherically symmetric, only I_{00} is non-zero, and the potential U does not depend on the angles (θ, ϕ). When the body is axially symmetric about the z axis, only the moments with $m = 0$ are non-zero, and U is independent of the azimuthal angle ϕ. Notice that each $I_{\ell m}$ scales as MR^ℓ, with R denoting a characteristic length scale of the body. The mass multipole moments introduced here are in a close correspondence with the charge multipole moments defined in electromagnetism; in fact, our definitions are virtually identical to those adopted by J.D. Jackson (1998) in his famous textbook *Classical Electrodynamics*.

Axially symmetric bodies

A body is axially symmetric when its mass density is invariant under a rotation about a symmetry axis. The condition can apply exactly to an idealized body, or it can apply as a good approximation to a realistic body (the Sun, for example). Taking the z-direction to be aligned with the symmetry axis, we find that the mass density is independent of the azimuthal angle ϕ, and it follows that the only non-vanishing multipole moments are those with $m = 0$. It is conventional to express the moments in terms of dimensionless quantities J_ℓ defined by

$$J_\ell := -\sqrt{\frac{4\pi}{2\ell + 1}} \frac{I_{\ell 0}}{MR^\ell}\,, \tag{1.143}$$

in which the characteristic length scale R is chosen to be the body's equatorial radius. This definition is adopted, for example, in *Allen's Astrophysical Quantities*, a repository of useful information about most areas of astrophysics.

The gravitational potential of an axially symmetric body can then be written in the form

$$U_{\text{ext}}(t, \boldsymbol{x}) = \frac{GM}{r}\left[1 - \sum_{\ell=2}^{\infty} J_\ell \left(\frac{R}{r}\right)^\ell P_\ell(\cos\theta)\right]. \tag{1.144}$$

The dominant term in the sum is provided by the dimensionless quadrupole moment J_2, and this is frequently expressed in terms of the body's *principal moments of inertia*, defined by

$$I_1(t) \equiv A(t) := \int \rho(t, \boldsymbol{x})(y^2 + z^2)\, d^3x\,, \tag{1.145a}$$

$$I_2(t) \equiv B(t) := \int \rho(t, \boldsymbol{x})(x^2 + z^2)\, d^3x\,, \tag{1.145b}$$

$$I_3(t) \equiv C(t) := \int \rho(t, \boldsymbol{x})(x^2 + y^2)\, d^3x\,. \tag{1.145c}$$

We have

$$\begin{aligned} J_2 &= -\frac{1}{MR^2}\int \rho r^2 \left[\frac{3\cos^2\theta - 1}{2}\right] d^3x \\ &= \frac{1}{MR^2}\int \rho \left[\frac{r^2 - 3z^2}{2}\right] d^3x \\ &= \frac{1}{MR^2}\int \rho \Big[x^2 + y^2 - \tfrac{1}{2}(x^2 + z^2) - \tfrac{1}{2}(y^2 + z^2)\Big] d^3x \\ &= \frac{C - \frac{1}{2}A - \frac{1}{2}B}{MR^2} \\ &= \frac{C - A}{MR^2}\,. \end{aligned} \tag{1.146}$$

In the last step we used the fact that $A = B$ for an axially symmetric body.

1.5.3 Symmetric tracefree tensors

We next turn to an alternative decomposition of the gravitational potential that involves tensorial combinations of the unit vector $\boldsymbol{n} := \boldsymbol{x}/r$ instead of spherical harmonics. Each tensor that we shall construct from $\boldsymbol{n}$ will have the property of being symmetric under the exchange of any two of its indices, and of being tracefree in any pair of indices; these tensors are known as *symmetric tracefree tensors*, or *STF tensors*. The decompositions in STF tensors and spherical harmonics both involve building blocks that consist of irreducible representations of the rotation group labelled by a multipole index ℓ. The decomposition in spherical harmonics relies on spherical polar coordinates, and keeps the polar angles (θ, ϕ) segregated from the radial coordinate r. The decomposition in STF tensors relies on the original Cartesian coordinates, which are all put on an equal footing. In our experience we have found that it is helpful to be conversant in both languages; some applications are best handled with spherical harmonics, and some are most easily treated with STF tensors.

Taylor expansion of the external potential

We return to the integral representation of the gravitational potential U,

$$U(t, \boldsymbol{x}) = G\int \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|}\, d^3x'\,, \tag{1.147}$$

and consider a field point $\boldsymbol{x}$ that lies outside the matter distribution. With $|\boldsymbol{x}'| < |\boldsymbol{x}|$, we carry out a Taylor expansion of $|\boldsymbol{x} - \boldsymbol{x}'|^{-1}$ in powers of $\boldsymbol{x}'$:

$$\frac{1}{|\boldsymbol{x} - \boldsymbol{x}'|} = \frac{1}{r} - x'^j \partial_j \left(\frac{1}{r}\right) + \frac{1}{2} x'^j x'^k \partial_j \partial_k \left(\frac{1}{r}\right) - \cdots \tag{1.148a}$$

$$= \frac{1}{r} - x'^j \partial_j \left(\frac{1}{r}\right) + \frac{1}{2} x'^{jk} \partial_{jk} \left(\frac{1}{r}\right) - \cdots \tag{1.148b}$$

$$= \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} x'^L \partial_L \left(\frac{1}{r}\right) . \tag{1.148c}$$

Once more we adopt the Einstein summation convention and sum over repeated indices in an expression like $x'^j x'^k \partial_j \partial_k r^{-1}$. In the second line we introduce a condensed notation in which an expression like x^{jkn} stands for the product $x^j x^k x^n$, and ∂_{jkn} stands for $\partial_j \partial_k \partial_n$. In the third line we introduce an even more compact *multi-index* notation, in which an uppercase index such as L represents a collection of ℓ individual indices. Thus, x^L stands for $x^{j_1 j_2 \cdots j_\ell}$, ∂_L stands for $\partial_{j_1 j_2 \cdots j_\ell}$, and $x^L \partial_L$ involves a summation over all ℓ pairs of repeated indices.

Substituting Eq. (1.148) into Eq. (1.147) gives

$$U_{\text{ext}}(t, \boldsymbol{x}) = G \sum_{\ell=0}^{\infty} \frac{(-1)^\ell}{\ell!} I^{\langle L \rangle} \partial_{\langle L \rangle} \left(\frac{1}{r}\right) , \tag{1.149}$$

with

$$I^{\langle L \rangle}(t) := \int \rho(t, \boldsymbol{x}') x'^{\langle L \rangle} \, d^3x' \tag{1.150}$$

defining a set of *STF multipole moments* for the mass distribution. The STF label, and the meaning of the angular brackets around the multi-index L, will be explained shortly. For the time being we may note that a comparison between Eq. (1.149) and Eq. (1.140) reveals a close relationship between $Y_{\ell m}/r^{\ell+1}$ and $\partial_{\langle L \rangle} r^{-1}$, and another between $I_{\ell m}$ and $I^{\langle L \rangle}$. The correspondence will be made precise below.

STF combinations

We next compute the derivatives of r^{-1} that appear in Eq. (1.149). We make repeated use of the identities

$$\partial_j r = n_j , \tag{1.151a}$$

$$\partial_j n_k = \partial_k n_j = \frac{1}{r} \left(\delta_{jk} - n_j n_k \right) , \tag{1.151b}$$

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where $\boldsymbol{n} := \boldsymbol{x}/r$ is a unit radial vector imagined to be expressed in terms of the Cartesian coordinates (x, y, z). We obtain

$$\partial_j r^{-1} = -n_j r^{-2}, \tag{1.152a}$$

$$\partial_{jk} r^{-1} = \bigl(3n_j n_k - \delta_{jk}\bigr) r^{-3}, \tag{1.152b}$$

$$\partial_{jkn} r^{-1} = -\Bigl[15 n_j n_k n_n - 3\bigl(n_j \delta_{kn} + n_k \delta_{jn} + n_n \delta_{jk}\bigr)\Bigr] r^{-4}, \tag{1.152c}$$

and so on; it is understood that $r \neq 0$ in these operations. We observe that the tensors on the right-hand side are all symmetric under an exchange of any two indices, and that they all vanish when a trace is taken over any pair of indices (which means that the indices within the pair are made equal and summed over); these are all examples of STF tensors. These properties are inherited from the definitions on the left-hand side: a tensor such as $\partial_{jkn} r^{-1}$ is necessarily symmetric because the partial derivatives commute with each other, and it is necessarily tracefree because, for example, $\delta^{jk} \partial_{jkn} r^{-1} = \nabla^2 \partial_n r^{-1} = \partial_n \nabla^2 r^{-1} = 0$. We conclude that each tensor in the collection $\partial_L r^{-1}$ is an STF tensor, a property that we can emphasize by enclosing L between angular brackets. More generally, any STF tensor will be distinguished with angular brackets; we shall write, for example, $A^{\langle jkn \rangle}$ for an STF tensor of rank 3, and $A^{\langle L \rangle}$ for an STF tensor of rank ℓ.

Conventionally, STF products of vectors such as n^j are obtained by beginning with the "raw" products $n^j n^k \cdots$ and then removing all traces, maintaining symmetry on all indices. Explicit examples are

$$n^{\langle jk \rangle} = n^j n^k - \frac{1}{3} \delta^{jk}, \tag{1.153a}$$

$$n^{\langle jkn \rangle} = n^j n^k n^n - \frac{1}{5}\bigl(\delta^{jk} n^n + \delta^{jn} n^k + \delta^{kn} n^j\bigr), \tag{1.153b}$$

$$\begin{aligned} n^{\langle jknp \rangle} = {} & n^j n^k n^n n^p - \frac{1}{7}\bigl(\delta^{jk} n^n n^p + \delta^{jn} n^k n^p + \delta^{jp} n^k n^n + \delta^{kn} n^j n^p \\ & + \delta^{kp} n^j n^n + \delta^{np} n^j n^k\bigr) + \frac{1}{35}\bigl(\delta^{jk}\delta^{np} + \delta^{jn}\delta^{kp} + \delta^{jp}\delta^{kn}\bigr). \end{aligned} \tag{1.153c}$$

For example $n^{\langle jkn \rangle}$ is tracefree because $\delta_{jk} n^{\langle jkn \rangle} = n^n - \frac{1}{5}(3n^n + n^n + n^n) = 0$, $\delta_{jn} n^{\langle jkn \rangle} = 0$, and $\delta_{kn} n^{\langle jkn \rangle} = 0$.

The general formula for such STF products is

$$\begin{aligned} n^{\langle j_1 j_2 \cdots j_\ell \rangle} = {} & \sum_{p=0}^{[\ell/2]} (-1)^p \frac{\ell!(2\ell - 2p - 1)!!}{(\ell - 2p)!(2\ell - 1)!!(2p)!!} \\ & \times \delta^{(j_1 j_2} \delta^{j_3 j_4} \ldots \delta^{j_{2p-1} j_{2p}} n^{j_{2p+1}} n^{j_{2p+2}} \cdots n^{j_\ell)}, \end{aligned} \tag{1.154}$$

in which $[\ell/2]$ is the largest integer not larger than $\ell/2$, equal to $\ell/2$ when ℓ is an even number and to $(\ell - 1)/2$ when ℓ is odd; all ℓ indices are enclosed within round brackets, which indicates the symmetrization operation defined in Box 1.4. In a more compact notation we have

$$n^{\langle L\rangle} = \sum_{p=0}^{[\ell/2]} (-1)^p \frac{(2\ell-2p-1)!!}{(2\ell-1)!!} \Big[\delta^{2P} n^{L-2P} + \text{sym}(q) \Big], \tag{1.155}$$

where δ^{2P} stands for a product of p Kronecker deltas (with indices running from j_1 to j_{2p}), n^{L-2P} stands for a product of $\ell - 2p$ unit vectors (with indices running from j_{2p+1} to j_ℓ), and "sym(q)" denotes all distinct terms arising from permuting indices; the total number of terms within the square brackets is equal to $q := \ell!/[(\ell-2p)!(2p)!!]$.

Note that the tensor $n^{\langle jk\rangle}$ contains five independent components; the number would be six for a general symmetric tensor, but the vanishing trace removes one component from the total count. This number matches the five values of m that belong to $\ell = 2$. Similarly, the tensor $n^{\langle jkn\rangle}$ contains seven independent components, and this matches the seven values of m that belong to $\ell = 3$. It can be shown that in general, $n^{\langle L\rangle}$ contains $2\ell+1$ independent components, and this is also the number of integers in the interval between $-\ell$ and $+\ell$.

Comparing Eqs. (1.152) and (1.153) we find that $\partial_j r^{-1} = -n_j r^{-2}$, $\partial_{jk} r^{-1} = 3n_{\langle jk\rangle} r^{-3}$, and $\partial_{jkn} r^{-1} = -15 n_{\langle jkn\rangle} r^{-4}$. The general rule can be obtained by induction:

$$\partial_L r^{-1} = \partial_{\langle L\rangle} r^{-1} = (-1)^\ell (2\ell-1)!! \, \frac{n_{\langle L\rangle}}{r^{\ell+1}} . \tag{1.156}$$

We may now return to Eq. (1.149), which can be expressed as

$$U_{\text{ext}}(t, \boldsymbol{x}) = G \sum_{\ell=0}^{\infty} \frac{(2\ell-1)!!}{\ell!} \, I^{\langle L\rangle} \frac{n_{\langle L\rangle}}{r^{\ell+1}} , \tag{1.157}$$

and explain the reason for the angular brackets on $I^{\langle L\rangle}$. In the preceding step, displayed in Eq. (1.148c), we expressed $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ as a sum of terms $x'^L \partial_L r^{-1}$, and substitution into Eq. (1.147) returned U as a sum of terms $I^L \partial_L r^{-1}$, with $I^L := \int \rho' x'^L \, d^3x'$ denoting the "raw" multipole moments. In view of Eq. (1.155), however, I^L differs from $I^{\langle L\rangle}$ by a sum of terms involving Kronecker deltas, and these automatically give zero when multiplied by the tracefree $\partial_L r^{-1}$. As a result, we find that $I^L \partial_L r^{-1} = I^{\langle L\rangle} \partial_{\langle L\rangle} r^{-1}$, and that U_{ext} can indeed be expressed as in Eq. (1.149).

It is worthwhile to display the main rule by which we were able to reach this conclusion: whenever an arbitrary tensor A^L multiplies an STF tensor $B_{\langle L\rangle}$, the outcome is

$$A^L B_{\langle L\rangle} = A^{\langle L\rangle} B_{\langle L\rangle}, \tag{1.158}$$

where $A^{\langle L\rangle}$ is the tensor obtained from A^L by complete symmetrization and removal of all traces.

STF identities

The STF tensors $n^{\langle L\rangle}$ satisfy a number of helpful identities, including

$$n_{\langle L\rangle}n^{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!}, \tag{1.159a}$$

$$n_j n^{\langle jL\rangle} = \frac{\ell+1}{2\ell+1}n^{\langle L\rangle}, \tag{1.159b}$$

$$n_{\langle L\rangle}n^{\langle jL\rangle} = \frac{(\ell+1)!}{(2\ell+1)!!}n^j. \tag{1.159c}$$

Other identities involve a second unit vector $\boldsymbol{n}'$:

$$n'_{\langle L\rangle}n^{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!}P_\ell(\mu), \tag{1.160a}$$

$$n'_{\langle L\rangle}n^{\langle jL\rangle} = \frac{\ell!}{(2\ell+1)!!}\left[\frac{dP_{\ell+1}}{d\mu}n^j - \frac{dP_\ell}{d\mu}n'^j\right], \tag{1.160b}$$

where $\mu := \boldsymbol{n}\cdot\boldsymbol{n}'$.

To establish these identities we begin with Eq. (1.160a), and write its left-hand side as $n'_L n^{\langle L\rangle}$ after invoking the rule of Eq. (1.158). We substitute Eq. (1.155) for $n^{\langle L\rangle}$ and perform the index contractions. For each value of p in the sum we find that $2p$ factors of $\boldsymbol{n}'$ multiply Kronecker deltas, returning unity, while the remaining $\ell-2p$ vectors multiply an $\boldsymbol{n}$, returning $\mu^{\ell-2p}$. Because all q terms are equal to each other, we get

$$n'_{\langle L\rangle}n^{\langle L\rangle} = \sum_{p=0}^{[\ell/2]}(-1)^p\frac{\ell!(2\ell-2p-1)!!}{(\ell-2p)!(2\ell-1)!!(2p)!!}\mu^{\ell-2p}. \tag{1.161}$$

Making use of the identities $(2p)!! = 2^p p!$ and $(2\ell-2p-1)!! = (2\ell-2p)!/[2^{\ell-p}(\ell-p)!]$, we find that this is also

$$n'_{\langle L\rangle}n^{\langle L\rangle} = \frac{\ell!}{(2\ell-1)!!}\frac{1}{2^\ell}\sum_{p=0}^{[\ell/2]}(-1)^p\frac{(2\ell-2p)!}{p!(\ell-p)!(\ell-2p)!}\mu^{\ell-2p}, \tag{1.162}$$

and the sum (together with the prefactor of $2^{-\ell}$) is recognized as a representation of the Legendre polynomial $P_\ell(\mu)$. We have recovered Eq. (1.160a), and we notice that Eq. (1.159a) is a special case with $\boldsymbol{n}'=\boldsymbol{n}$, $\mu=1$, and $P_\ell(\mu)=1$.

To establish Eq. (1.159b) we observe that the product $n_j n^{\langle jL\rangle}$ is necessarily STF in the indices contained in L, and that it must therefore be proportional to $n^{\langle L\rangle}$. The constant of proportionality can be determined by making use of Eq. (1.159a) in the form of $n_{\langle jL\rangle}n^j n^L = (\ell+1)!/(2\ell+1)!!$; the end result is Eq. (1.159b). The identity of Eq. (1.159c) can be established by similar means.

To establish Eq. (1.160b) we observe that $n'_{\langle L\rangle}n^{\langle jL\rangle}$ must be a vector constructed from n^j and n'^j. We may write it as $(\ell+1)!(an^j+bn'^j)/(2\ell+1)!!$ and work to determine the coefficients a and b; the factor of $(\ell+1)!/(2\ell+1)!!$ is inserted for convenience.

Using Eqs. (1.159b) and (1.160a) it is easy to see that a and b must satisfy the equations $a + b\mu = P_\ell(\mu)$ and $a\mu + b = P_{\ell+1}(\mu)$. The solutions are $a = (\mu P_{\ell+1} - P_\ell)/(\mu^2 - 1)$ and $b = (\mu P_\ell - P_{\ell+1})/(\mu^2 - 1)$, and these can be re-expressed as $a = (\ell + 1)^{-1} dP_{\ell+1}/d\mu$ and $b = -(\ell + 1)^{-1} dP_\ell/d\mu$ by exploiting the recurrence relations satisfied by the Legendre polynomials. The end result is Eq. (1.160b).

Correspondence with spherical harmonics

We are now ready to reveal the correspondence between the STF tensors $n^{\langle L\rangle}$ and the spherical harmonics $Y_{\ell m}(\theta, \phi)$. It comes about when $\boldsymbol{n}$ is expressed as

$$\boldsymbol{n} = [\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta], \tag{1.163}$$

and when it is recognized that $n^{\langle L\rangle}$ is a set of functions of θ and ϕ that can be decomposed in spherical harmonics, as in Eq. (1.118). The decomposition, however, involves a single value of ℓ (instead of a sum over all values), and the $2\ell + 1$ values of m that belong to this ℓ. The reason is that $n^{\langle L\rangle}$ is not just any function; as we show in Box 1.6, it is an *eigenfunction* of the (angular piece of the) Laplacian operator: $r^2\nabla^2 n^{\langle L\rangle} = -\ell(\ell + 1)n^{\langle L\rangle}$. Because $n^{\langle L\rangle}$ satisfies the same eigenvalue equation as $Y_{\ell m}(\theta, \phi)$, the expansion must be of the form

$$n^{\langle L\rangle} := N_\ell \sum_{m=-\ell}^{\ell} \mathscr{Y}^{\langle L\rangle}_{\ell m} Y_{\ell m}(\theta, \phi), \qquad N_\ell := \frac{4\pi\,\ell!}{(2\ell + 1)!!}, \tag{1.164}$$

where $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ is a constant STF tensor that satisfies $\mathscr{Y}^{\langle L\rangle}_{\ell,-m} = (-1)^m \mathscr{Y}^{*\langle L\rangle}_{\ell m}$, and N_ℓ is a normalization constant chosen for future convenience. In Box 1.5 we display a few members of $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ for selected values of ℓ.

When we multiply Eq. (1.164) by $Y^*_{\ell m'}$ and integrate over the whole sphere, we obtain

$$\mathscr{Y}^{\langle L\rangle}_{\ell m} = \frac{1}{N_\ell} \int n^{\langle L\rangle} Y^*_{\ell m}(\theta, \phi)\, d\Omega. \tag{1.165}$$

When we next multiply this expression by $n'_{\langle L\rangle}$ and make use of Eq. (1.160a), we get

$$\mathscr{Y}^{\langle L\rangle}_{\ell m} n'_{\langle L\rangle} = \frac{1}{N_\ell} \int n^{\langle L\rangle} n'_{\langle L\rangle} Y^*_{\ell m}(\theta, \phi)\, d\Omega = \frac{2\ell + 1}{4\pi} \int P_\ell(\mu) Y^*_{\ell m}(\theta, \phi)\, d\Omega. \tag{1.166}$$

When, finally, we insert the addition theorem of Eq. (1.137) with $\cos\gamma = \mu = \boldsymbol{n}\cdot\boldsymbol{n}'$ and perform the integration, we arrive at

$$Y_{\ell m}(\theta, \phi) = \mathscr{Y}^{*\langle L\rangle}_{\ell m} n_{\langle L\rangle}, \tag{1.167}$$

a decomposition of the spherical harmonics in STF tensors. Equation (1.167) is the inverse of Eq. (1.164), and N_ℓ was chosen so as to make the overall factor on the right-hand side of Eq. (1.167) equal to unity.

Box 1.6 Proof that $r^2\nabla^2 n^{\langle L\rangle} = -\ell(\ell+1)n^{\langle L\rangle}$

To prove that $n^{\langle L\rangle}$ satisfies the eigenvalue equation, it is convenient to work instead with the scalar field $\psi := A_{\langle L\rangle}n^{\langle L\rangle}$, in which $A_{\langle L\rangle}$ is an arbitrary STF tensor of constant elements. Because this tensor is arbitrary, it can be chosen so as to single out any particular element of $n^{\langle L\rangle}$, and it becomes sufficient to prove that ψ itself satisfies the eigenvalue equation.

We write the scalar field as $\psi = A_{\langle L\rangle}x^L/r^\ell$ and differentiate it once with respect to x^j. Because $A_{\langle L\rangle}$ is completely symmetric, we have that $A_{\langle L\rangle}\partial_j x^L = A_{\langle k_1k_2\cdots k_\ell\rangle}\partial_j x^{k_1k_2\cdots k_\ell} = \ell A_{\langle jk_2\cdots k_\ell\rangle}x^{k_2\cdots k_\ell} = \ell A_{\langle jL-1\rangle}x^{L-1}$. Combining this with $\partial_j r^{-\ell} = -\ell\, n_j r^{-(\ell+1)}$, we find that

$$\partial_j\psi = \ell A_{\langle jL-1\rangle}\frac{x^{L-1}}{r^\ell} - \ell A_{\langle L\rangle}\frac{x^L}{r^{\ell+1}}\, n_j\,.$$

Proceeding along the same lines to compute the second derivative, and noting that $A_{\langle jjL-2\rangle} = 0$ and $\partial_j n_j = 2/r$, we finally arrive at

$$r^2\nabla^2\psi = -\ell(\ell+1)\psi\,.$$

Because $\psi = A_{\langle L\rangle}n^{\langle L\rangle}$ and $A_{\langle L\rangle}$ is arbitrary, this proves that $n^{\langle L\rangle}$ itself satisfies the eigenvalue equation.

With the connection between STF tensors and spherical harmonics displayed in Eqs. (1.164) and (1.167), it is easy to show that the multipole moments of Eqs. (1.139) and (1.150) are related by

$$I_{\ell m} = \mathscr{Y}^{\langle L\rangle}_{\ell m} I_{\langle L\rangle}\,, \tag{1.168a}$$

$$I^{\langle L\rangle} = \frac{4\pi\,\ell!}{(2\ell+1)!!}\sum_{m=-\ell}^{\ell}\mathscr{Y}^{*\langle L\rangle}_{\ell m} I_{\ell m}\,. \tag{1.168b}$$

With these results, the equivalence of Eqs. (1.140) and (1.149) is immediately established.

The foregoing results give rise to another identity, which will be required in Chapter 6. We rewrite Eq. (1.164) in terms of a different ℓ' and different direction $\boldsymbol{n}'$ and get

$$n'^{\langle L'\rangle} = N_{\ell'}\sum_{m'=-\ell'}^{\ell'}\mathscr{Y}^{\langle L'\rangle}_{\ell' m'} Y_{\ell' m'}(\theta',\phi')\,. \tag{1.169}$$

Multiplying by $Y^*_{\ell m}(\theta',\phi')$, integrating over $d\Omega'$, and using the orthonormality of the spherical harmonics, we next obtain

$$\int Y^*_{\ell m}(\theta',\phi')n'^{\langle L'\rangle}\,d\Omega' = \delta_{\ell\ell'}N_\ell\,\mathscr{Y}^{\langle L\rangle}_{\ell m}\,. \tag{1.170}$$

If we now multiply each side by $Y_{\ell m}(\theta,\phi)$, sum over m, and insert Eq. (1.164), we finally obtain

$$\sum_{m=-\ell}^{\ell} Y^*_{\ell m}(\theta,\phi)\int Y_{\ell m}(\theta',\phi')n'^{\langle L'\rangle}\,d\Omega' = \delta_{\ell\ell'}\,n^{\langle L\rangle}\,; \tag{1.171}$$

this is the required identity.

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Angular averages

There will be many occasions, in this book, when we need to calculate the average of a quantity $\psi(\theta, \phi)$ over the surface of a sphere:

$$\langle\langle \psi \rangle\rangle := \frac{1}{4\pi} \int \psi(\theta, \phi)\, d\Omega \,. \tag{1.172}$$

Of particular interest are the spherical average of products $n^j n^k n^n \cdots$ of radial vectors. These are easily computed using the fact that the average of the STF tensor $n^{\langle jkn\cdots\rangle}$ must be zero; this property follows directly from Eq. (1.164) and the identity $\int Y_{\ell m}(\theta, \phi)\, d\Omega = 0$ (unless $\ell = 0$). In this way we obtain

$$\langle\langle n^j \rangle\rangle = 0 \,, \tag{1.173a}$$

$$\langle\langle n^j n^k \rangle\rangle = \frac{1}{3}\delta^{jk} \,, \tag{1.173b}$$

$$\langle\langle n^j n^k n^n \rangle\rangle = 0 \,, \tag{1.173c}$$

$$\langle\langle n^j n^k n^n n^p \rangle\rangle = \frac{1}{15}\big(\delta^{jk}\delta^{np} + \delta^{jn}\delta^{kp} + \delta^{jp}\delta^{kn}\big) \,, \tag{1.173d}$$

and so on. These results can also be established directly, by recognizing that the tensorial structure on the right-hand side is uniquely determined by the complete symmetry of the left-hand side and the fact that δ^{jk} is the only available geometrical object. The numerical coefficient can then be determined by taking traces; for example, $1 = \delta_{jk}\delta_{np}\langle\langle n^j n^k n^n n^p \rangle\rangle = \frac{1}{15}(9 + 3 + 3)$, and this confirms that the numerical coefficient must indeed be $\frac{1}{15}$.

The general expression for such angular averages can be shown to be given by

$$\langle\langle n^L \rangle\rangle = \frac{1}{(\ell + 1)!!}\Big[\delta^L + \text{sym}(q)\Big], \tag{1.174}$$

when ℓ is an even number, and $\langle\langle n^L \rangle\rangle = 0$ when ℓ is odd; we use the same notation as in Eq. (1.155), in which δ^L stands for a product of $\ell/2$ Kronecker deltas, and sym(q) denotes all distinct terms obtained by permuting indices; the total number of terms within the square brackets is equal to $q = (\ell - 1)!!$.

1.6 Motion of extended fluid bodies

In this final section of Chapter 1 we examine a specific kind of fluid system, one that consists of separated blobs of fluids surrounded by vacuum. Each blob is called a "body," and the bodies are imagined to be in orbital motion around one another, the motion governed by the mutual gravitational attractions. Examples of such systems abound in the universe: we may be speaking of a binary system of main-sequence stars, or of a solar system of (gaseous) planets orbiting a central sun. (Although the discussion below relies on the bodies being made up of a perfect fluid, the final results apply just as well to solid bodies such as Earth-like planets.)

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1.6.1 From fluid configurations to isolated bodies

Consider, therefore, a fluid configuration that is broken up into a collection of separated bodies. The configuration is characterized by two length scales, the typical size R of each body, and the typical separation r between bodies. We assume that $R < r$, so that the bodies are indeed separated, and surrounded by vacuum regions of space; our discussion excludes contact binaries, in which two stars share a common envelope. Each body is assumed to be isolated, in the sense that no matter is ejected from, nor accreted by, the body. Our discussion therefore excludes stars with strong stellar winds, such as Wolf–Rayet stars, which can lose mass at a dynamically significant rate. It excludes also interacting binary-star systems, for which a transfer of mass from one star to the other can have important effects on the orbital motion. The assumption, however, is a good one for most binary stars, and also for our solar system, in which the effects of the solar wind and its associated mass loss on planetary motions can be safely neglected, at least over the time scales we might be interested in.

The formalism developed in this section applies to separated and isolated bodies, and it actually relies on the stronger inequality $R \ll r$, which states that the inter-body separation is very large compared with the extension of each body. The strong inequality comes with a number of important consequences that we now describe.

The external, inter-body dynamics is governed by mutual gravitational interactions, and it proceeds on an orbital time scale given approximately by $T_{\rm orb} \sim (r^3/Gm)^{1/2}$, where m is the mass of a typical body. The internal, intra-body dynamics is governed instead by hydrodynamical processes, and it proceeds on an internal time scale given approximately by $T_{\rm int} \sim (G\rho)^{-1/2} \sim (R^3/Gm)^{1/2}$. So $T_{\rm int} \ll T_{\rm orb}$ when $R \ll r$, and the internal and external dynamics take place over widely separated time scales. A consequence of this fact is that the internal and external dynamics are largely decoupled from each other. It is possible, for example, for an orbiting body to be in a state of (approximate) hydrodynamic equilibrium even when external gravitational forces are applied to it, and for the orbital motion to be (approximately) independent of the details of the internal state.

As we shall see in Chapter 2, the strong inequality $R \ll r$ also implies that the tidal interaction between bodies is small. When other sources of deformation (such as rotation) are also small, the bodies can be taken to be nearly spherical. In such circumstances the gravitational field of each body can be well approximated by a multipole expansion of the sort developed in Sec. 1.5.2.

The (approximate) decoupling of the internal and external dynamics, and the (approximate) near-spherical nature of the bodies, produce a substantial simplification of the mathematical description of the inter-body motion. Instead of the original description, which involved the fine-grained fluid variables $(\rho, p, \boldsymbol{v})$, the orbital motion can be described with a smaller set of coarse-grained variables that characterize each body as a whole; these include the body's mass, center-of-mass position, spin angular momentum, and a number of multipole moments which encapsulate the required details of the internal dynamics.

Our main goal in this section is to accomplish this coarse-grained description of the external dynamics. In Chapter 2 we shall return to the internal dynamics and describe the internal structure of self-gravitating bodies. It is important to bear in mind that while the internal and external problems are approximately decoupled from one another, they are not fully decoupled: Some aspects of the external dynamics (such as the tidal coupling between bodies) depend on internal processes, and aspects of the internal dynamics (such as tidal deformations) depend on the orbital motion; ultimately and fundamentally the internal and external problems are informed by each other. A simple example is provided by the Earth–Moon system. The Moon raises tides (both solid and oceanic) on the Earth, and these depend on the Moon's orbital position; the tidal deformation of the Earth then modifies its own gravitational potential, and this affects the orbit of the Moon.

1.6.2 Center-of-mass variables

We consider a fluid system that is broken up into a number N of separated and isolated bodies, in the sense provided in Sec. 1.6.1. Each body is assigned a label $A = 1, 2, \ldots, N$, and each body occupies a volume V_A bounded by a closed surface S_A. The mass density ρ is assumed to be equal to ρ_A inside V_A, and zero in the vacuum region between bodies. The fluid dynamics inside each body is governed by the Euler and continuity equations – Eqs. (1.23) and (1.25) – and the gravitational potential U is given everywhere by Eq. (1.12).

The total mass of body A is given by

$$m_A := \int_A \rho(t, \boldsymbol{x})\, d^3x \,, \tag{1.175}$$

where the domain of integration is a fixed region of space that extends slightly beyond the volume V_A; it is sufficiently small that it contains no other body, but sufficiently large that it continues to contain body A as it moves about in a small interval of time dt. It is easy to show, using the techniques developed in Sec. 1.4.3, that m_A is time-independent: $dm_A/dt = 0$.

We define the center-of-mass position of body A (see Box 1.7) by

$$\boldsymbol{r}_A(t) := \frac{1}{m_A} \int_A \rho(t, \boldsymbol{x}) \boldsymbol{x}\, d^3x \,, \tag{1.176}$$

and we similarly define the center-of-mass velocity and acceleration by

$$\boldsymbol{v}_A(t) := \frac{1}{m_A} \int_A \rho(t, \boldsymbol{x}) \boldsymbol{v}\, d^3x \,, \tag{1.177}$$

and

$$\boldsymbol{a}_A(t) := \frac{1}{m_A} \int_A \rho(t, \boldsymbol{x}) \frac{d\boldsymbol{v}}{dt}\, d^3x \,. \tag{1.178}$$

The integration techniques of Sec. 1.4.3 imply that

$$\boldsymbol{v}_A = \frac{d\boldsymbol{r}_A}{dt} \,, \qquad \boldsymbol{a}_A = \frac{d\boldsymbol{v}_A}{dt} \,. \tag{1.179}$$

In addition to these variables we introduce

$$I_A^{\langle L \rangle}(t) := \int_A \rho(t, \boldsymbol{x})\, (x - r_A)^{\langle L \rangle}\, d^3x \,, \tag{1.180}$$

the STF multipole moments of body A, which refer to its center-of-mass position $\boldsymbol{r}_A(t)$; note that the dipole moment $I_A^j = \int_A \rho(x - r_A)^j\, d^3x$ vanishes by virtue of Eq. (1.176). These

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definitions, and the results of Eq. (1.179), form the core of the coarse-grained formulation of the external problem. Instead of the original fluid variables $(\rho, p, \boldsymbol{v})$, the equations of motion of each body will be written in terms of $\boldsymbol{r}_A(t)$ and $I_A^{\langle L\rangle}(t)$; instead of functions of time *and* space, the formulation involves functions of time only.

Box 1.7 Is the center-of-mass unique?

The definition of the center-of-mass position proposed in Eq. (1.176) is not unique. For example, we could equally well propose the alternative definition $\boldsymbol{r}_A := m_A^{-1} \int_A (\rho^2/\langle\rho\rangle)\boldsymbol{x}\, d^3x$, in which $\langle\rho\rangle$ is the mean density inside body A; this would in general produce a different position for the center-of-mass. The main requirements for a sensible definition of center-of-mass are that it be located somewhere inside the body (it should not wander too far off), that it be useful and convenient, and that it be used consistently in all developments. Once these requirements are satisfied, the freedom of choice is unlimited, and ultimately the most important aspect is the matter of convenience.

It is important to bear in mind that the choice carries no physical consequence: there is no measurable way to determine the true position of the center-of-mass. When astronomers track planets using telescopes, they track the geometrical center of the image, or the location of the edge as it occults a star or the Sun. When they bounce radar or laser beams off planets, they determine the distance between the beam emitter and a point on the surface. When they determine the motion of a planet by tracking a satellite orbiting around it, they perform a complicated reduction of the orbital data to determine what they call a "normal point," the effective center-of-mass of the planet that controls the spacecraft's orbit. Given the shape and orientation of the body, one only needs to be able to go back and forth between the center-of-mass, as conventionally defined, and the place on the surface that is actually being located, or to the trajectory of an orbiting satellite.

The choice of center-of-mass proposed in Eq. (1.176) is useful and convenient because $\boldsymbol{r}_A(t)$ remains at rest or moves uniformly when the body is not subjected to a net force, and because Eqs. (1.179) have a nice structure. As we shall see, the equations of motion for extended bodies based on Eq. (1.176) are about as simple as they can be (although we admit that simplicity is a subjective notion).

The lack of uniqueness becomes even more acute in special and general relativity, because there are now many different densities that could be involved in a definition of center-of-mass: density of rest mass alone, or density of rest mass plus other forms of internal energy, such as kinetic, thermodynamic, or even gravitational binding energy. One could even include contributions from the gravitational potential energy provided by other bodies in the system, so that the center-of-mass position of a body might depend on the location of its neighbors. In addition, the very act of integrating the vector $\boldsymbol{x}$ over the body is problematic in relativity, because of ambiguities associated with the choice of reference frame. The problem is most serious in general relativity, because of the additional ambiguities associated with the choice of coordinate system. And finally, deep and subtle issues arise in the definition of center-of-mass for spinning bodies in special and general relativity. There is a vast literature devoted to attempts to define *the* center-of-mass, most of it extremely formal, and little of it of practical use. In the relativistic part of this book we will accept the arbitrariness of the center-of-mass, and adopt definitions that are as useful and convenient as we can make them, even if they are not provided with complete relativistic rigor.

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The equations of motion of body A are obtained by inserting Euler's equation (1.23) within Eq. (1.178). The term involving the pressure gradient is easily disposed of: it integrates to zero after invoking Gauss's theorem, because $p = 0$ on the boundary of the domain of integration. What remains is

$$m_A \boldsymbol{a}_A = \int_A \rho \boldsymbol{\nabla} U \, d^3x \,. \tag{1.181}$$

Summing over all bodies and making use of Eq. (1.56), we find that

$$\sum_{A=1}^{N} m_A \boldsymbol{a}_A = \int_{\text{all space}} \rho \boldsymbol{\nabla} U \, d^3x = 0 \,. \tag{1.182}$$

This is a statement of Newton's third law, and a confirmation that

$$\boldsymbol{R} := \frac{1}{m} \sum_A m_A \boldsymbol{r}_A \,, \tag{1.183}$$

the *barycenter* of the N-body system, moves uniformly with a constant velocity $\boldsymbol{V}$; here $m := \sum_A m_A$ is the total mass of the system.

1.6.3 Internal and external potential

The gravitational potential that appears in Eq. (1.181) is produced in part by body A, and in part by all the remaining bodies. To distinguish between these contributions we decompose U as

$$U = U_A + U_{\neg A} \,, \tag{1.184}$$

with

$$U_A(t, \boldsymbol{x}) = G \int_A \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x' \tag{1.185}$$

denoting the piece produced by body A – the internal potential – and

$$U_{\neg A}(t, \boldsymbol{x}) = \sum_{B \neq A} G \int_B \frac{\rho(t, \boldsymbol{x}')}{|\boldsymbol{x} - \boldsymbol{x}'|} \, d^3x' \tag{1.186}$$

denoting the piece produced by the remaining bodies – the external potential.

When we insert Eq. (1.184) within Eq. (1.181) we find that the internal potential makes no contribution to the equations of motion. This comes as a consequence of the identity

$$\int_A \rho \boldsymbol{\nabla} U_A \, d^3x = 0 \,, \tag{1.187}$$

which is the statement of Eq. (1.56) applied to body A instead of the entire fluid system; the identity is established by following the same sequence of steps that led to the derivation of Eq. (1.56). As a result of this simplification, the equations of motion become

$$m_A \boldsymbol{a}_A = \int_A \rho \boldsymbol{\nabla} U_{\neg A} \, d^3x \,, \tag{1.188}$$

and they involve only the external potential of Eq. (1.186).

1.6.4 Taylor expansion of the external potential

At this stage we incorporate our assumption that the bodies are well separated, so that $R_A \ll r_{AB}$, with R_A denoting the characteristic size of body A, and $r_{AB} := |\boldsymbol{r}_A - \boldsymbol{r}_B|$ denoting the typical separation between bodies. The variable of integration $\boldsymbol{x}$ in Eq. (1.188) ranges over the small scale R_A, and because the external potential $U_{\neg A}$ varies over the much larger scale r_{AB}, it is appropriate to express it as the Taylor expansion

$$U_{\neg A}(t, \boldsymbol{x}) = U_{\neg A}(t, \boldsymbol{r}_A) + (x - r_A)^j \partial_j U_{\neg A}(t, \boldsymbol{r}_A) + \frac{1}{2}(x - r_A)^{jk} \partial_{jk} U_{\neg A}(t, \boldsymbol{r}_A) + \cdots , \tag{1.189}$$

in which the potential is evaluated at $\boldsymbol{x} = \boldsymbol{r}_A$ after differentiation. In a compact multi-index notation, this is

$$U_{\neg A}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x - r_A)^L \partial_L U_{\neg A}(t, \boldsymbol{r}_A) . \tag{1.190}$$

Because the external potential satisfies Laplace's equation $\nabla^2 U_{\neg A} = 0$ within the volume occupied by body A, its partial derivatives form a STF tensor, and using the rule of Eq. (1.158) we can write

$$(x - r_A)^L \partial_L U_{\neg A} = (x - r_A)^L \partial_{\langle L\rangle} U_{\neg A} = (x - r_A)^{\langle L\rangle} \partial_{\langle L\rangle} U_{\neg A} . \tag{1.191}$$

This gives rise to our final expression for the external potential,

$$U_{\neg A}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x - r_A)^{\langle L\rangle} \partial_L U_{\neg A}(t, \boldsymbol{r}_A) , \tag{1.192}$$

from which we have removed the redundant angular brackets on $\partial_{\langle L\rangle}$.

The gradient of the external potential is then given by

$$\partial_j U_{\neg A}(t, \boldsymbol{x}) = \sum_{\ell=0}^{\infty} \frac{1}{\ell!}(x - r_A)^{\langle L\rangle} \partial_{jL} U_{\neg A}(t, \boldsymbol{r}_A) , \tag{1.193}$$

and substitution within Eq. (1.188) returns

$$m_A a_A^j = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} I_A^{\langle L\rangle}(t)\, \partial_{jL} U_{\neg A}(t, \boldsymbol{r}_A) \tag{1.194}$$

after involving the definition of Eq. (1.180). At this stage we have the equations of motion expressed in terms of the multipole moments of body A and partial derivatives of the external potential $U_{\neg A}$ evaluated at $\boldsymbol{x} = \boldsymbol{r}_A$.

To proceed we must work on the external potential. We return to its definition of Eq. (1.186), and for each body B within the sum we express the variable of integration $\boldsymbol{x}'$ as

$$\boldsymbol{x}' = \boldsymbol{r}_B(t) + \bar{\boldsymbol{x}}' , \tag{1.195}$$

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with $\bar{\boldsymbol{x}}'$ describing a displacement from B's center-of-mass. Each term in the sum is of the form

$$G\int_B \frac{\rho(t, \boldsymbol{r}_B + \bar{\boldsymbol{x}}')}{|\boldsymbol{x} - \boldsymbol{r}_B - \bar{\boldsymbol{x}}'|}\, d^3\bar{x}', \tag{1.196}$$

in which the new integration variable $\bar{\boldsymbol{x}}'$ ranges over the small scale R_B. Because $\boldsymbol{x} - \boldsymbol{r}_B$ is of the order of the much larger scale r_{AB}, it is appropriate to express the denominator as the Taylor expansion

$$\begin{aligned}|\boldsymbol{x} - \boldsymbol{r}_B - \bar{\boldsymbol{x}}'|^{-1} &= |\boldsymbol{x} - \boldsymbol{r}_B|^{-1} - \bar{x}'^p \partial_p |\boldsymbol{x} - \boldsymbol{r}_B|^{-1} + \frac{1}{2}\bar{x}'^{pq}\partial_{pq}|\boldsymbol{x} - \boldsymbol{r}_B|^{-1} + \cdots \\ &= \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!} \bar{x}'^{L'} \partial_{L'} |\boldsymbol{x} - \boldsymbol{r}_B|^{-1} \\ &= \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!} \bar{x}'^{\langle L'\rangle} \partial_{L'} |\boldsymbol{x} - \boldsymbol{r}_B|^{-1}.\end{aligned} \tag{1.197}$$

Making the substitution in Eq. (1.196) and invoking once more the definition of Eq. (1.180), we arrive at

$$U_{\neg A}(t, \boldsymbol{x}) = G \sum_{B\neq A} \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!} I_B^{\langle L'\rangle} \partial_{L'} |\boldsymbol{x} - \boldsymbol{r}_B|^{-1}, \tag{1.198}$$

an expression for the external potential that involves the multipole moments of each external body B.

We may now take the additional derivatives that are required in Eq. (1.194) and evaluate the result at $\boldsymbol{x} = \boldsymbol{r}_A$. The result is

$$\partial_{jL} U_{\neg A}(t, \boldsymbol{r}_A) = G \sum_{B\neq A} \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell'!} I_B^{\langle L'\rangle} \partial_{jLL'} \left(\frac{1}{r_{AB}}\right), \tag{1.199}$$

with $r_{AB} := |\boldsymbol{r}_A - \boldsymbol{r}_B|$ denoting the inter-body distance. The notation requires some care in interpretation: by $\partial_{jLL'} r_{AB}^{-1}$ we mean "take $\ell' + \ell + 1$ derivatives of $|\boldsymbol{x} - \boldsymbol{r}_B|^{-1}$ with respect to $\boldsymbol{x}$ and evaluate the result at $\boldsymbol{x} = \boldsymbol{r}_A$." To simplify this and eliminate the risk of confusion, we choose to express the operation in the equivalent form

$$\partial^A_{jLL'}\left(\frac{1}{r_{AB}}\right),$$

which now means "take $\ell' + \ell + 1$ derivatives of r_{AB}^{-1} with respect to $\boldsymbol{r}_A$."

1.6.5 Equations of motion for isolated bodies

The hard work is over. Substitution of Eq. (1.199) into Eq. (1.194) returns our final expression for the center-of-mass acceleration of body A. We obtain

$$m_A a_A^j = G \sum_{B\neq A} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle L\rangle} I_B^{\langle L'\rangle} \partial^A_{jLL'} \left(\frac{1}{r_{AB}}\right). \tag{1.200}$$

This equation, as it stands, is exact. Because each multipole moment $I_A^{\langle L\rangle}$ scales as $m_A R_A^\ell$, each term in the sum scales as

$$\frac{Gm_A m_B}{r_{AB}^2}\left(\frac{R_A}{r_{AB}}\right)^\ell \left(\frac{R_B}{r_{AB}}\right)^{\ell'},$$

and the assumption that $R_A \ll r_{AB}$ ensures that each term gets progressively smaller; the equation is exact, but it is most useful as a starting point for an approximation scheme. For many applications involving a small ratio R_A/r_{AB}, the sums can be safely truncated after just a few terms. For other applications, however, a large number of terms may be required. An example is the motion of a satellite in a low Earth orbit, which is sensitive to many of Earth's multipole moments; in the satellite geodesy project GRACE (Gravity Recovery and Climate Experiment), multipole moments up to $\ell \sim 360$ have been measured.

To rewrite Eq. (1.200) in a friendlier form we first isolate the early terms in the sums over ℓ and ℓ', noting that the monopole moment of body A is simply its mass, $I_A = m_A$, and that its dipole moment vanishes, $I_A^j = 0$, by virtue of the definition of the center-of-mass. We next split the sums into a piece that is linear in the higher multipole moments (coming from the terms $\ell = 0$, $\ell' \geq 2$ or $\ell' = 0$, $\ell \geq 2$) and another piece that involves products of the moments. This yields

$$a_A^j = G\sum_{B\neq A}\left\{-\frac{m_B}{r_{AB}^2}n_{AB}^j + \sum_{\ell=2}^\infty \frac{1}{\ell!}\left[(-1)^\ell I_B^{\langle L\rangle} + \frac{m_B}{m_A}I_A^{\langle L\rangle}\right]\partial^A_{jL}\left(\frac{1}{r_{AB}}\right) + \frac{1}{m_A}\sum_{\ell=2}^\infty\sum_{\ell'=2}^\infty \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle L\rangle} I_B^{\langle L'\rangle}\partial^A_{jLL'}\left(\frac{1}{r_{AB}}\right)\right\}, \tag{1.201}$$

where $\boldsymbol{n}_{AB} := \boldsymbol{r}_{AB}/r_{AB}$ is a unit vector that points from body B to body A. This expression implies that $\sum_A m_A \boldsymbol{a}_A = \boldsymbol{0}$, a statement that was already established in Eq. (1.182).

The equations displayed in Eq. (1.201) form a complete set of equations of motion for the N bodies once their masses and multipole moments as functions of time are specified. They can be integrated once the initial position and velocity of each body are given. The equations, however, provide an *incomplete description* of the physical system, because they do not determine the time-evolution of the multipole moments; these will, in general, depend on the details of the internal structure of each body and the motion of the remaining bodies. The multipole moments capture the remaining coupling between the internal and external problems (as discussed in Sec. 1.6.1), and additional information must be supplied in order to turn Eq. (1.201) into a closed system of equations of motion. We shall return to this issue in Chapter 2.

The multipole moments of a perfectly spherical body vanish, $I_A^{\langle L\rangle} = 0$ for $\ell \neq 0$, and when all the bodies are spherical we find that Eq. (1.201) reduces to the familiar set of point-mass equations of motion,

$$a_A^j = -\sum_{B\neq A}\frac{Gm_B}{r_{AB}^2}n_{AB}^j\,. \tag{1.202}$$

When the bodies are not spherical, we observe that the motion of body A is affected by the distortion of the gravitational potential caused by the deformation of the other bodies; this

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influence is described by the terms in Eq. (1.201) that involve $I_B^{\langle L\rangle}$. It is affected also by the coupling of its own non-spherical mass distribution to gradients of the monopole field of each external body; this influence is described by the terms in Eq. (1.201) that are linear in $I_A^{\langle L\rangle}$. And finally, it is affected by couplings between its own multipole moments and those of the external bodies, as described by the last line in Eq. (1.201). This last effect is analogous to the dipole–dipole coupling in electrodynamics, except for the fact that there is no dipole moment in gravitation; the leading effect comes from a quadrupole–quadrupole interaction. The presence of terms involving $I_A^{\langle L\rangle}$ in the equations of motion implies that the motion of a body can depend on its internal structure, by virtue of its finite size and the non-spherical coupling of its mass distribution to the external gravitational field. This observation does not constitute a violation of the weak equivalence principle; a violation would imply a dependence on internal structure that remains even when the bodies have a negligible size.

1.6.6 Conserved quantities

In Sec. 1.4.3 we showed that the total mass, momentum, energy, and angular momentum of a fluid configuration are conserved as a consequence of the fluid's dynamics. We recall that the total momentum $\boldsymbol{P}$ is defined by Eq. (1.54), the total energy E is defined by Eq. (1.67), and the total angular momentum $\boldsymbol{J}$ is defined by Eq. (1.78). These quantities continue to be conserved when the fluid configuration describes a system of isolated bodies, and in this section we derive expressions for the total momentum, energy, and angular momentum of an N-body system.

We begin with the definition of total momentum, $\boldsymbol{P} = \int \rho\boldsymbol{v}\, d^3x$, in which the integral over all space may be written as a sum of integrals extending over each body. In each integral we decompose $\boldsymbol{v}$ as $(\boldsymbol{v} - \boldsymbol{v}_A) + \boldsymbol{v}_A$, with the first term describing a velocity relative to the center-of-mass of body A. The integral of $\rho(\boldsymbol{v} - \boldsymbol{v}_A)$ vanishes by virtue of the definition of the center-of-mass – refer back to Eq. (1.177) – and the second integral yields $m_A\boldsymbol{v}_A$. The final result is

$$\boldsymbol{P} = \sum_A m_A \boldsymbol{v}_A\,; \tag{1.203}$$

as expected, the total momentum is a simple sum of individual momenta.

A similar computation returns

$$\boldsymbol{J} = \sum_A \bigl(\boldsymbol{S}_A + m_A \boldsymbol{r}_A \times \boldsymbol{v}_A\bigr) \tag{1.204}$$

for the total angular momentum, where

$$\boldsymbol{S}_A := \int_A \rho\bigl(\boldsymbol{x} - \boldsymbol{r}_A\bigr) \times \bigl(\boldsymbol{v} - \boldsymbol{v}_A\bigr)\, d^3x \tag{1.205}$$

is the intrinsic angular momentum of body A – its spin. We see that the total angular momentum is a simple sum of individual spin and orbital angular momenta.

The calculation of the total energy involves a computation of the kinetic energy $\mathcal{T} = \frac{1}{2}\int \rho v^2\, d^3x$, the gravitational potential energy $\Omega = -\frac{1}{2}\int \rho U\, d^3x$, and the internal

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(thermodynamic) energy $E^{\rm int} = \int \epsilon\, d^3x$. For the kinetic and internal energies we immediately get

$$\mathcal{T} = \sum_A \Big(\mathcal{T}_A + \frac{1}{2} m_A v_A^2\Big) \tag{1.206}$$

and

$$E^{\rm int} = \sum_A E_A^{\rm int}, \tag{1.207}$$

where

$$\mathcal{T}_A := \frac{1}{2}\int_A \rho \big|\boldsymbol{v} - \boldsymbol{v}_A\big|^2\, d^3x \tag{1.208}$$

is the internal kinetic energy of body A, while

$$E_A^{\rm int} := \int_A \epsilon\, d^3x \tag{1.209}$$

is its own thermodynamic energy.

The computation of Ω requires more work. We first return to Eq. (1.184) and decompose the gravitational potential into internal and external pieces, $U = U_A + U_{\neg A}$. This gives rise to

$$\Omega = \sum_A \bigg(\Omega_A - \frac{1}{2}\int_A \rho U_{\neg A}\, d^3x\bigg), \tag{1.210}$$

in which

$$\Omega_A := -\frac{1}{2}\int_A \rho U_A\, d^3x \tag{1.211}$$

is the internal gravitational potential energy of body A. To evaluate the second term we follow the strategy of Sec. 1.6.4 and express $U_{\neg A}(t, \boldsymbol{x})$ as a Taylor expansion about $\boldsymbol{x} = \boldsymbol{r}_A$. Using the expression of Eq. (1.192), we obtain

$$-\frac{1}{2}\int_A \rho U_{\neg A}\, d^3x = -\frac{1}{2} m_A U_{\neg A}\big(t, \boldsymbol{r}_A\big) - \frac{1}{2}\sum_{\ell=2}^{\infty} \frac{1}{\ell!} I_A^{\langle L\rangle} \partial_L U_{\neg A}\big(t, \boldsymbol{r}_A\big), \tag{1.212}$$

in which $I_A^{\langle L\rangle}(t)$ are the multipole moments of body A, as defined by Eq. (1.180); there is no $\ell = 1$ term in the sum because $I_A^j = 0$ by virtue of the definition of the center-of-mass. In the remaining steps we express the external potential as an expansion in inverse powers of $r_{AB} := |\boldsymbol{r}_A - \boldsymbol{r}_B|$, as in Eq. (1.198). After some simplification we arrive at our final expression, which is recognized below as the collection of terms involving the nested sums over pairs of bodies.

Collecting results, we find that the total energy of a system of isolated bodies is given by

$$\begin{aligned} E = &\sum_A E_A + \sum_A \frac{1}{2} m_A v_A^2 - \frac{1}{2}\sum_A \sum_{B\neq A} \frac{G m_A m_B}{r_{AB}} \\ &- \frac{1}{2}\sum_A \sum_{B\neq A} \sum_{\ell=2}^{\infty} \frac{1}{\ell!}\Big[(-1)^\ell G m_A I_B^{\langle L\rangle} + G m_B I_A^{\langle L\rangle}\Big]\partial_L^A \left(\frac{1}{r_{AB}}\right) \\ &- \frac{1}{2}\sum_A \sum_{B\neq A} \sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} G I_A^{\langle L\rangle} I_B^{\langle L'\rangle} \partial_{LL'}^A \left(\frac{1}{r_{AB}}\right), \end{aligned} \tag{1.213}$$

where $E_A := \mathcal{T}_A + \Omega_A + E_A^{\rm int}$ is the self-energy of body A. The manipulations following Eq. (1.67) can immediately be adapted to each body, and the conclusion is that each self-energy is individually conserved. Because the term $\sum_A E_A$ merely contributes an irrelevant constant to E, it can be safely removed from a conventional accounting of total energy, which holds that the energy should vanish when $v_A \to 0$ and $r_{AB} \to \infty$. In the final analysis we shall retain only the center-of-mass kinetic energies and the mutual interaction energies in Eq. (1.213).

When the bodies are spherical, so that $I_A^{\langle L\rangle} = 0$ for $\ell \neq 0$, the total energy reduces to

$$E = \sum_A \frac{1}{2} m_A v_A^2 - \frac{1}{2}\sum_A \sum_{B\neq A} \frac{G m_A m_B}{r_{AB}}, \tag{1.214}$$

the familiar expression for a system of point masses.

1.6.7 Equations of motion for binary systems

We next specialize the general discussion of this section to a system of two bodies. Our binary system consists of a first body of mass m_1 and multipole moments $I_1^{\langle L\rangle}$ at a position $\boldsymbol{r}_1$, and a second body of mass m_2 and multipole moments $I_2^{\langle L\rangle}$ at a position $\boldsymbol{r}_2$. The total mass of the binary system is $m := m_1 + m_2$. In place of $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ it is useful to work with the *barycenter position*

$$\boldsymbol{R} := \frac{m_1}{m}\boldsymbol{r}_1 + \frac{m_2}{m}\boldsymbol{r}_2, \tag{1.215}$$

and the *relative separation*

$$\boldsymbol{r} := \boldsymbol{r}_1 - \boldsymbol{r}_2. \tag{1.216}$$

This vector was denoted $\boldsymbol{r}_{12}$ in preceding subsections, and we simplify other notations in a similar way by defining

$$r := |\boldsymbol{r}|, \qquad \boldsymbol{n} := \boldsymbol{r}/r. \tag{1.217}$$

It is useful to note that $\boldsymbol{r}_{21} = -\boldsymbol{r}$, $\boldsymbol{n}_{21} = -\boldsymbol{n}$, and that $r_{21} = r$. In addition to the relative separation we also introduce the relative velocity $\boldsymbol{v} := d\boldsymbol{r}/dt = \boldsymbol{v}_1 - \boldsymbol{v}_2$ and the relative acceleration

$$\boldsymbol{a} := \frac{d^2\boldsymbol{r}}{dt^2} = \boldsymbol{a}_1 - \boldsymbol{a}_2. \tag{1.218}$$

Solving for $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$, we find that

$$\boldsymbol{r}_1 = \boldsymbol{R} + \frac{m_2}{m}\boldsymbol{r}\,, \qquad \boldsymbol{r}_2 = \boldsymbol{R} - \frac{m_1}{m}\boldsymbol{r}\,. \tag{1.219}$$

The motion of the binary system is determined when $\boldsymbol{R}(t)$ and $\boldsymbol{r}(t)$ are both known as functions of time. The motion of the barycenter is uniform: as we saw at the end of Sec. 1.6.2, it is described by $\boldsymbol{R}(t) = \boldsymbol{R}(0) + \boldsymbol{V}t$, where $\boldsymbol{V} := \boldsymbol{P}/m$ is a constant velocity vector. The relative motion is governed by

$$\begin{aligned} a^j = {} & -\frac{Gm}{r^2}n^j + Gm\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[\frac{I_1^{\langle L\rangle}}{m_1} + (-1)^\ell \frac{I_2^{\langle L\rangle}}{m_2}\right]\partial_{jL}\left(\frac{1}{r}\right) \\ & + Gm\sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty}\frac{(-1)^{\ell'}}{\ell!\ell'!}\frac{I_1^{\langle L\rangle}}{m_1}\frac{I_2^{\langle L'\rangle}}{m_2}\partial_{jLL'}\left(\frac{1}{r}\right), \end{aligned} \tag{1.220}$$

an effective one-body equation that can easily be obtained from Eq. (1.201). The derivation relies on the fact that $\partial^2_j r^{-1}_{21} = -\partial^1_j r^{-1}_{12} := -\partial_j r^{-1}$; in this notation ∂_j indicates partial differentiation with respect to the Cartesian coordinate r^j associated with the relative separation $\boldsymbol{r}$. From Eq. (1.213) we find that the total energy (excluding self-energies) of a two-body system is given by

$$\begin{aligned} E = {} & \frac{1}{2}mV^2 + \frac{1}{2}\mu v^2 - \frac{G\mu m}{r} \\ & - G\mu m\sum_{\ell=2}^{\infty}\frac{1}{\ell!}\left[\frac{I_1^{\langle L\rangle}}{m_1} + (-1)^\ell\frac{I_2^{\langle L\rangle}}{m_2}\right]\partial_L\left(\frac{1}{r}\right) \\ & - G\mu m\sum_{\ell=2}^{\infty}\sum_{\ell'=2}^{\infty}\frac{(-1)^{\ell'}}{\ell!\ell'!}\frac{I_1^{\langle L\rangle}}{m_1}\frac{I_2^{\langle L'\rangle}}{m_2}\partial_{LL'}\left(\frac{1}{r}\right), \end{aligned} \tag{1.221}$$

in which $\mu := m_1 m_2/m$ is the system's *reduced mass*.

As a specialization of these equations we assume that the multipole moments of one of the bodies, say body 1, are negligible. This simplification would apply, for example, to a planet orbiting the Sun (the planet has negligible moments), to a satellite orbiting the Earth (the satellite has negligible moments), or to a non-rotating black hole or neutron star orbiting a normal star (the compact object has negligible moments). In this case we find that the relative acceleration simplifies to

$$a^j = -\frac{Gm}{r^2}n^j + Gm\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\frac{I_2^{\langle L\rangle}}{m_2}\partial_{jL}\left(\frac{1}{r}\right), \tag{1.222}$$

and the total energy becomes

$$E = \frac{1}{2}mV^2 + \frac{1}{2}\mu v^2 - \frac{G\mu m}{r} - G\mu m\sum_{\ell=2}^{\infty}\frac{(-1)^\ell}{\ell!}\frac{I_2^{\langle L\rangle}}{m_2}\partial_L\left(\frac{1}{r}\right). \tag{1.223}$$

For a planet orbiting the Sun, or a spacecraft orbiting the Earth, m_1 is much smaller than m_2, and $m/m_2 \simeq 1$. In this case the equations describe the motion of a spherical body in the multipole field of a heavy, central object. In situations involving comparable masses, however, such as a black hole or neutron star orbiting a normal star, the ratio m/m_2 could be substantially larger than unity, reflecting the fact that the motion of both bodies can be strongly affected by the multipole moments of body 2.

Specializing even further, we now take body 2 to be symmetric about an axis aligned with the unit vector $\boldsymbol{e}$. The symmetry requires the body's multipole moments $I^{\langle L\rangle}$ to be proportional to the STF tensor $e^{\langle L\rangle}$, so that $I^{\langle L\rangle} = \alpha_\ell e^{\langle L\rangle}$ for some coefficient α_ℓ. We wish to relate this to the dimensionless multipole moments J_ℓ introduced in Eq. (1.143). To achieve this we align the z-direction with the vector $\boldsymbol{e}$ and invoke Eqs. (1.112), (1.167), and (1.168). After some algebra we obtain $\alpha_\ell = -mR^\ell J_\ell$, so that

$$I_2^{\langle L\rangle} = -m_2 R_2^\ell (J_\ell)_2\, e_2^{\langle L\rangle}; \tag{1.224}$$

to indicate that all quantities refer to body 2 we have inserted the label "2" on all quantities (such as mass, radius, symmetry axis, and multipole moments) that appear in Eq. (1.224).

Equation (1.222) can then be written as

$$a^j = -\frac{Gm}{r^2}\left[n^j - \sum_{\ell=2}^{\infty}\frac{(2\ell+1)!!}{\ell!}(J_\ell)_2\left(\frac{R_2}{r}\right)^\ell e_2^{\langle L\rangle} n_{\langle jL\rangle}\right], \tag{1.225}$$

after making use of Eq. (1.156) to express the derivatives of r^{-1} in terms of STF products of the vector $\boldsymbol{n}$; the product $e_2^{\langle L\rangle} n_{\langle jL\rangle}$ could be further simplified by invoking Eq. (1.160b). With similar manipulations we can show that the total energy becomes

$$E = \frac{1}{2}mV^2 + \frac{1}{2}\mu v^2 - \frac{G\mu m}{r}\left[1 - \sum_{\ell=2}^{\infty}(J_\ell)_2\left(\frac{R_2}{r}\right)^\ell P_\ell(\boldsymbol{e}_2\cdot\boldsymbol{n})\right]; \tag{1.226}$$

to arrive at this result we have made use of Eq. (1.159a) to express $e_2^{\langle L\rangle} n_{\langle L\rangle}$ in terms of Legendre polynomials.

For modestly deformed bodies, and for sufficiently large separations, the $\ell = 2$ term dominates in both Eq. (1.225) and Eq. (1.226). In this case the relative acceleration becomes

$$\boldsymbol{a} = -\frac{Gm}{r^2}\left\{\boldsymbol{n} - \frac{3}{2}(J_2)_2\left(\frac{R_2}{r}\right)^2\left\{\left[5(\boldsymbol{e}_2\cdot\boldsymbol{n})^2 - 1\right]\boldsymbol{n} - 2(\boldsymbol{e}_2\cdot\boldsymbol{n})\boldsymbol{e}_2\right\}\right\}; \tag{1.227}$$

the expression involves the total mass $m := m_1 + m_2$, and it applies to a binary system of arbitrary mass ratio. This equation, the specialization of Eq. (1.220) to a spherical body moving in the monopole and quadrupole field of an axisymmetric body, is the foundation for the study of a number of important phenomena, including the effect of the solar quadrupole moment on the orbit of Mercury, and the precession of the planes of Earth-orbiting satellites. We shall return to these applications in Chapter 3.

1.6.8 Spin dynamics

As we saw back in Eq. (1.205), the *spin angular momentum* of body A is defined by

$$\boldsymbol{S}_A(t) := \int_A \rho(t,\boldsymbol{x})(\boldsymbol{x} - \boldsymbol{r}_A)\times(\boldsymbol{v} - \boldsymbol{v}_A)\, d^3x, \tag{1.228}$$

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and it refers to its center-of-mass position $\boldsymbol{r}_A$ and velocity $\boldsymbol{v}_A$. In terms of components we use the permutation symbol ϵ^{jpq} to describe the vectorial product, and we write

$$S_A^j(t) := \epsilon^{jpq} \int_A \rho(x - r_A)^p (v - v_A)^q \, d^3x \,. \tag{1.229}$$

We wish to find an equation of motion for $\boldsymbol{S}_A(t)$, and we shall proceed by following the general method outlined in Secs. 1.6.2, 1.6.3, and 1.6.4.

We begin by differentiating Eq. (1.229) with respect to t. Exploiting once again the techniques developed in Sec. 1.4.3, we find that

$$\begin{aligned} \frac{dS_A^j}{dt} &= \epsilon^{jpq} \int_A \rho(v - v_A)^p (v - v_A)^q \, d^3x + \epsilon^{jpq} \int_A \rho(x - r_A)^p (dv/dt - a_A)^q \, d^3x \\ &= \epsilon^{jpq} \int_A \rho(x - r_A)^p \frac{dv^q}{dt} \, d^3x \,, \end{aligned} \tag{1.230}$$

where we have used the fact that $\int_A \rho(x - r_A)^p \, d^3x = 0$ by virtue of the definition of the center-of-mass position. In this we insert Euler's equation (1.23) and obtain

$$\frac{dS_A^j}{dt} = \epsilon^{jpq} \int_A \rho(x - r_A)^p \partial_q U \, d^3x - \epsilon^{jpq} \int_A (x - r_A)^p \partial_q p \, d^3x \,. \tag{1.231}$$

The second term, involving the pressure p, can be integrated by parts; after discarding the boundary term we are left with $\epsilon^{jpq} \delta_{pq} \int_A p \, d^3x$, which vanishes identically. We have obtained

$$\frac{dS_A^j}{dt} = \epsilon^{jpq} \int_A \rho(x - r_A)^p \partial_q U \, d^3x \,, \tag{1.232}$$

and in this we insert the decomposition of the gravitational potential in terms of internal and external pieces, as in Eq. (1.184). It is easy to show that the contribution from the internal potential,

$$\epsilon^{jpq} \int_A \rho \, x^p \partial_q U_A \, d^3x - \epsilon^{jpq} r_A^p \int_A \rho \, \partial_q U_A \, d^3x \,,$$

is in fact zero. The first term vanishes by virtue of Eq. (1.80) (applied to body A instead of the entire N-body system), and the second term vanishes thanks to Eq. (1.187). The evolution of the spin is therefore governed by

$$\frac{dS_A^j}{dt} = \epsilon^{jpq} \int_A \rho(x - r_A)^p \partial_q U_{\neg A} \, d^3x \,, \tag{1.233}$$

which involves the gravitational potential $U_{\neg A}(t, \boldsymbol{x})$ produced by the bodies external to A.

At this stage we import Eq. (1.193), which provides an expression for $\partial_q U_{\neg A}(t, \boldsymbol{x})$ as a Taylor expansion in powers of $\boldsymbol{x} - \boldsymbol{r}_A$. Inserting this within Eq. (1.233), we obtain

$$\begin{aligned} \frac{dS_A^j}{dt} &= \epsilon^{jpq} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} I_A^{pL} \partial_{qL} U_{\neg A}(t, \boldsymbol{r}_A) \\ &= \epsilon^{jpq} \sum_{\ell=0}^{\infty} \frac{1}{\ell!} I_A^{\langle pL \rangle} \partial_{qL} U_{\neg A}(t, \boldsymbol{r}_A) \,. \end{aligned} \tag{1.234}$$

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In the second line we allowed ourselves to enclose the indices pL within angular brackets, recognizing that the difference between $I_A^{\langle pL\rangle}$ and I_A^{pL} involves a number of Kronecker deltas that either (i) force indices contained in ∂_{qL} to be equal, giving zero when acting on $U_{\neg A}$, or (ii) force the derivative operator to be of the form ∂_{pqL-1}, which vanishes when multiplied by ϵ^{jpq}. We next import Eq. (1.199) and obtain

$$\frac{dS_A^j}{dt} = G\epsilon^{jpq} \sum_{B\neq A} \sum_{\ell=0}^{\infty} \sum_{\ell'=0}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle pL\rangle} I_B^{\langle L'\rangle} \partial^A_{\langle qLL'\rangle} \left(\frac{1}{r_{AB}}\right). \tag{1.235}$$

To write this in a friendlier form we observe that the terms with $\ell = 0$ make no contributions (because the dipole moment of body A vanishes), that the terms with $\ell' = 0$ involve m_B only, and that the terms with $\ell' = 1$ also make no contributions. We can therefore split the sum into two pieces, one linear in the moments of body A, and the other involving products of moments. We have

$$\begin{aligned} \frac{dS_A^j}{dt} &= G\epsilon^{jpq} \sum_{B\neq A} \sum_{\ell=1}^{\infty} \frac{1}{\ell!} m_B I_A^{\langle pL\rangle} \partial^A_{\langle qL\rangle} \left(\frac{1}{r_{AB}}\right) \\ &\quad + G\epsilon^{jpq} \sum_{B\neq A} \sum_{\ell=1}^{\infty} \sum_{\ell'=2}^{\infty} \frac{(-1)^{\ell'}}{\ell!\ell'!} I_A^{\langle pL\rangle} I_B^{\langle L'\rangle} \partial^A_{\langle qLL'\rangle} \left(\frac{1}{r_{AB}}\right), \end{aligned} \tag{1.236}$$

and this equation determines the behavior of each spin once the multipole moments and the center-of-mass motion of each body are specified.

We next specialize the discussion to an N-body system that consists of a spinning body A with non-vanishing multipole moments, and external bodies B with negligible multipole moments. In addition, we assume that body A is symmetric about an axis aligned with the unit vector $\boldsymbol{e}_A$. Under these conditions we have that

$$\boldsymbol{S}_A = S_A \boldsymbol{e}_A, \qquad S_A := |\boldsymbol{S}_A|, \tag{1.237}$$

and Eq. (1.224) implies that the body's multipole moments are given by

$$I_A^{\langle L\rangle} = -m_A R_A^\ell (J_\ell)_A \, e_A^{\langle L\rangle}. \tag{1.238}$$

This relation is inserted within Eq. (1.236), along with Eq. (1.156), and after some algebra we obtain

$$\frac{dS_A^j}{dt} = -\epsilon^{jpq} \sum_{B\neq A} \frac{Gm_A m_B}{r_{AB}} \sum_{\ell=1}^{\infty} (-1)^{\ell+1} \frac{(2\ell+1)!!}{\ell!} (J_{\ell+1})_A \left(\frac{R_A}{r_{AB}}\right)^{\ell+1} e_A^{\langle pL\rangle} n_{AB}^{\langle qL\rangle}. \tag{1.239}$$

This is simplified with the help of Eq. (1.160b), and we express the final result as

$$\frac{d\boldsymbol{S}_A}{dt} = -\sum_{B\neq A} \frac{Gm_A m_B}{r_{AB}} (\boldsymbol{e}_A \times \boldsymbol{n}_{AB}) \sum_{\ell=2}^{\infty} (-1)^\ell (J_\ell)_A \left(\frac{R_A}{r_{AB}}\right)^\ell \frac{dP_\ell}{d\mu_{AB}}, \tag{1.240}$$

in which $P_\ell(\mu_{AB})$ is a Legendre polynomial, and $\mu_{AB} := \boldsymbol{e}_A \cdot \boldsymbol{n}_{AB}$. This equation implies that the magnitude of the spin vector stays constant, because according to Eq. (1.237), $dS_A/dt = \boldsymbol{e}_A \cdot d\boldsymbol{S}_A/dt = 0$. And indeed, we observe that each term in the sum over B would give rise to a precession of $\boldsymbol{e}_A$ in the direction of $\boldsymbol{n}_{AB}$; after summation the precession is seen to take place in a direction given by a weighted average of all the vectors $\boldsymbol{n}_{AB}$.

As we shall see in Chapter 3, one notable consequence of Eq. (1.240) is the disturbance of the Earth's axis caused by the coupling of its equatorial bulge with the gravitational fields of the Sun and Moon. This leads to the famous precession of the equinoxes, with its cycle of approximately 26 000 years.

1.7 Bibliographical notes

The presentation of the basic equations of Newtonian gravity in Sec. 1.2 follows the standard treatment found in many undergraduate texts, including the venerable *Newtonian Mechanics* by French (1971). The theory, of course, was created in Newton's own *Principia*, which can be accessed in the superb English edition with extensive commentary by Cohen, Whitman, and Budenz (1999). The Eöt-Wash torsion balance experiment is described in Su *et al.* (1994) and Baessler *et al.* (1999).

The theory of Green's functions touched upon in Secs. 1.3 and 1.5 is developed systematically in many textbooks on mathematical methods, including the excellent *Mathematical Methods for Physicists* by Arfken, Weber, and Harris (2012).

The discussion of Sec. 1.4 relies on elements of fluid mechanics, thermodynamics, and statistical physics. Those are covered in many textbooks. An elegant and sophisticated development of fluid mechanics can be found in the classic *Fluid Mechanics* by Landau and Lifshitz (1987), and another useful resource is Kundu, Cohen, and Dowling (2011). A comprehensive presentation of thermodynamics and statistical physics can be found in Reif's *Fundamentals of Statistical and Thermal Physics*, now available in a new 2008 edition.

The development of multipole expansions to integrate Poisson's equation in Sec. 1.5 relies on the theory of spherical harmonics, a topic covered in most textbooks on mathematical methods. These developments are virtually identical to those related to the electrostatic potential, which are described in most textbooks on electromagnetism; the most comprehensive is the classic *Classical Electromagnetism* by Jackson (1998). The use of symmetric-tracefree tensors as substitutes for spherical harmonics was pioneered by Sachs (1961) and Pirani (1964); a systematic treatment can be found in Thorne (1980), and another useful resource is Damour and Iyer (1991). The citation from the *Principia* is taken from the Cohen, Whitman, and Budenz edition. The book *Allen's Astrophysical Quantities* is edited by Cox (2001).

An overview of the GRACE geodesy project, mentioned in Sec. 1.6, can be found at `www.csr.utexas.edu/grace/gravity/geodesy.html`.

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1.8 Exercises

1.1 Show explicitly that for a function $f(t, \boldsymbol{x}, \boldsymbol{x}')$,

$$\frac{\partial^2}{\partial t^2}\int \rho' f\, d^3x' = \int \rho'\left(\frac{\partial^2 f}{\partial t^2} + 2\boldsymbol{v}'\cdot\nabla'\frac{\partial f}{\partial t} + \frac{d\boldsymbol{v}'}{dt}\cdot\nabla' f + v'^j v'^k \partial_j\partial_k f\right) d^3x' .$$

1.2 Given the Newtonian potential $U(t, \boldsymbol{x})$, one can define a *superpotential* $X(t, \boldsymbol{x})$, a *superduperpotential* $Y(t, \boldsymbol{x})$, and another superlative potential $Z(t, \boldsymbol{x})$ that satisfy the equations

$$\nabla^2 X = 2U , \qquad \nabla^2 Y = 12X , \qquad \nabla^2 Z = 30Y .$$

Find explicit expressions for X, Y, and Z as integrals over the mass density $\rho(t, \boldsymbol{x}')$, assuming that ρ vanishes outside some finite region of space.

1.3 Using the expression for the superpotential X obtained in the preceding problem, show that

$$\frac{\partial^2}{\partial t^2}X(t, \boldsymbol{x}) = -\int \rho'\frac{d\boldsymbol{v}'}{dt}\cdot\frac{(\boldsymbol{x}-\boldsymbol{x}')}{|\boldsymbol{x}-\boldsymbol{x}'|}\, d^3x' + \int \frac{\rho'}{|\boldsymbol{x}-\boldsymbol{x}'|}\left\{v'^2 - \frac{[\boldsymbol{v}'\cdot(\boldsymbol{x}-\boldsymbol{x}')]^2}{|\boldsymbol{x}-\boldsymbol{x}'|^2}\right\} d^3x' .$$

1.4 Prove that

$$\int \rho(t, \boldsymbol{x}) x^j v^k\, d^3x = \frac{1}{2}\frac{dI^{jk}}{dt} + \frac{1}{2}\epsilon^{jkp}J^p ,$$

where I^{jk} is the quadrupole moment tensor of the mass distribution, and J^p is the total angular momentum.

1.5 Assuming that $T^{\alpha\beta} = 0$ far away from the system, use the equations of hydrodynamics in the form of $\partial_\beta T^{\alpha\beta} = 0$ to verify explicitly that the total mass M, momentum $\boldsymbol{P}$, and angular momentum $\boldsymbol{J}$ of an isolated system are all constant.

1.6 With the same assumptions as in the preceding problem, prove that a statement of the tensorial virial theorem is

$$\frac{d^2 I^{jk}}{dt^2} = \int T^{jk}\, d^3x ,$$

where $I^{jk} := \int T^{tt} x^j x^k\, d^3x$. Then show that with T^{jk} given by Eq. (1.96), the virial theorem takes the explicit form of Eq. (1.88).

1.7 Use the spherical-harmonic expansion of $|\boldsymbol{x}-\boldsymbol{x}'|^{-1}$ to verify that

$$U(t, r) = \frac{Gm(t, r)}{r} + 4\pi G\int_r^R \rho(t, r')r'dr'$$

for a spherical matter distribution.

1.8 Show explicitly that $\partial_{jknp} r^{-1} = 105 n^{\langle jknp\rangle}/r^5$.

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1.9 Show that the forms of $n^{\langle jk\rangle}$, $n^{\langle jkn\rangle}$, $n^{\langle jknp\rangle}$ given by Eq. (1.153) satisfy the general formula of Eq. (1.155).

1.10 Find $n^{\langle jknpq\rangle}$ by explicit construction.

1.11 Show that the internal gravitational potential of Eq. (1.129) can be expressed as

$$U = G\sum_{\ell=0}^{\infty}\frac{(-1)^\ell}{\ell!}\Big[q^{\langle L\rangle}(t,r)\,\partial_L r^{-1} + p^{\langle L\rangle}(t,r)\,x^{\langle L\rangle}\Big],$$

where

$$q^{\langle L\rangle}(t,r) := \int_0^r \rho(t,\boldsymbol{x}')x'^{\langle L\rangle}\,d^3x', \qquad p^{\langle L\rangle}(t,r) := \int_r^R \rho(t,\boldsymbol{x}')\partial_L r'^{-1}\,d^3x'.$$

In the integral defining $q^{\langle L\rangle}(t,r)$, the domain of integration is the region of space bounded by a sphere of radius $r := |\boldsymbol{x}|$. In the integral defining $p^{\langle L\rangle}(t,r)$, the domain of integration is the region of space bounded inwardly by a sphere of radius r, and outwardly by a sphere of arbitrary radius R that lies outside the distribution of matter.

1.12 For $\ell = 2$, 3, and 4, show explicitly that $n'^{\langle L\rangle}n^{\langle L\rangle} = [\ell!/(2\ell-1)!!]P_\ell(\mu)$, where $\mu := \boldsymbol{n}'\cdot\boldsymbol{n}$.

1.13 Fill in all the steps that are required to establish the STF identities of Eqs. (1.159) and (1.160).

1.14 If $\boldsymbol{e}$ and $\boldsymbol{n}$ are unit vectors, show that

$$e^{\langle qL\rangle}n^{\langle pL\rangle} = \frac{\ell!}{(2\ell+1)(2\ell+1)!!}\bigg[\delta^{pq}\frac{dP_\ell}{d\mu} - \big(e^pe^q + n^pn^q\big)\frac{d^2P_{\ell+1}}{d\mu^2} + e^{(p}n^{q)}\bigg(2\frac{d^2P_\ell}{d\mu^2} + (2\ell+1)\frac{dP_{\ell+1}}{d\mu}\bigg) + (2\ell+1)e^{[p}n^{q]}\frac{dP_{\ell+1}}{d\mu}\bigg],$$

where $\mu = \boldsymbol{e}\cdot\boldsymbol{n}$, and use this to verify Eq. (1.240). *Hint:* exploit the fact that $e^{\langle qL\rangle} \propto \partial_{qL}(1/R)$ and $n^{\langle pL\rangle}\partial_{pL}(1/r)$, where R and r are independent distance variables.

1.15 Determine the STF tensors $\mathscr{Y}^{\langle L\rangle}_{\ell m}$ for $\ell = 1$, $\ell = 2$, and $\ell = 3$, and thereby verify the results listed in Box 1.5.

1.16 Using the general equation of motion (1.201), show explicitly that $\sum_A m_A\boldsymbol{a}_A = \boldsymbol{0}$.